ISING MODELS AND SOLITON EQUATIONS

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1. Introduction

Since the original contributions /1-6/ of Onsager, Kaufman, Yang, and Lee, the two-dimensional Ising model has remained in the focus of attention to this day, simply because it is a unique laboratory both for theoretical and experimental research. It is a nontrivial many-body system for which many, (but most likely not all), properties can be evaluated exactly. From the early work, we know the zero-field free energy /1,2/, the long-range order (spontaneous magnetization) /3/, some information on the short-range order /4/ and the location of the singularities in the complex magnetic field plane /5,6/. Various other approaches have been invented, since, and further results for planar lattices other than quadratic and for correlation functions have been obtained. Most of the developments until the early seventies are presented in the textbooks of Green and Hurst /7/ and of McCoy and Wu /8/.

Then, in 1973, Wu, McCoy, Tracy, and Barouch /9,10/ discovered that in the so-called scaling limit towards the critical point, the scaled two-point function satisfies the Painlevé III equation (or Euclidean sinh-Gordon equation). Scaling limit results for n-point functions were given also /11-13/ and the Kyoto mathematicians Sato, Miwa, and Jimbo /14/ succeeded in deriving an integrable set of nonlinear partial difference equations for arbitrary temperatures on general planar lattices /15-18/, reducing to partial differential equations in the scaling limit. A review, also giving many further references to the Ising literature, is published by one of us in the proceedings of the previous symposium here /18/. Although the full details have not yet been presented, they are quite analogous to and can be easily reconstructed from those for the Ising chain in a transverse field /19/, see also the review /20/. One of the most striking results is that the two-point correlation at the critical point /16/ satisfies Hirota's discrete-time Toda lattice equation /21/. This does not require a lengthy derivation, but can instead be seen as an immediate application of a general version of the Wick theorem /19/ or, equivalently, an application of the compound Pfaffian theorem /17-19/. It is amusing to see that this last theorem existed before in the Ising literature as an exercise to the reader /22/, and was, in fact, over a hundred years old /23/; but to our knowledge it was never used before in

connection with correlation functions. It is a powerful tool /24/, leading to major shortcuts in the original methods /9-15/.

In this talk we shall start from Hirota's equation and derive several new results for the critical correlations, which should be compared with the work of Fisher /25,26/ and Wu /8,27/. They could not be obtained easily from the type of expansions generally used in the literature /8-14/ since these have zero radius of convergence at the critical point. Even though we shall be mainly concerned with the critical two-point functions here, it should be stressed that more general soliton-type difference equations have been derived for all temperatures and all multi-point correlation functions /15-18/ in the two-dimensional Ising model. Other, yet unknown equations, must govern the correlations in general integrable lattice models. We should mention at this point the recent successes of the conformal-invariance approach in the determination of critical exponents /28-32/ and, especially, critical correlations in the field theory limit /28-31/. In this approach the two-spin correlation function is predicted to be rotationally invariant and to decay with a power law. In our approach, however, we have obtained systematic corrections due to the underlying lattice, breaking the rotational invariance. We hope that further work on multi-spin correlations, will provide insight in how to generalize the concept of local conformal invariance to lattice models, also in the massive (noncritical) case. This, then, could give a comprehensive theory of correlation functions in two-dimensional solvable models.

After presenting our results, we shall show how these give the monomer-monomer correlation /33,34/ in the otherwise closest-packed dimer system on a regular square lattice, and correlations in the Ising model in a magnetic field $\frac{1}{2} i\pi kT$ /6,35/ or its dual, the fully-frustrated Ising model /36,37/. We shall conclude by comparing the present Euclidean case with its Minkowski counterpart in the time-continuum limit, alias the transverse Ising chain /19,38,39/.

2. Preliminaries and isotropic case

The two-dimensional Ising model on a regular square lattice is defined by the hamiltonian

$$-\beta_c \mathcal{H} = \sum_{M,N} (H_c \sigma_{MN}\sigma_{M,N+1} + V_c \sigma_{MN}\sigma_{M+1,N}) , \tag{1}$$

where $\sigma = \pm 1$, M and N are the vertical and horizontal coordinates, and H_c and V_c are the horizontal and vertical coupling constants multiplied with $\beta_c = 1/kT_c$, the inverse critical temperature. The condition of criticality is

$$\sinh(2H_c)\,\sinh(2V_c) = 1. \tag{2}$$

It is known for a long time that the critical spin-spin correlation function, given by the canonical average

$$C(M,N) \equiv \langle \sigma_{00}\sigma_{MN} \rangle , \tag{3}$$

reduces to a Cauchy determinant on the diagonal M=N, with the result /8/

$$C(M,M) = \left(\frac{2}{\pi}\right)^M \prod_{\ell=1}^{M} \left[1 - \frac{1}{4\ell^2}\right]^{\ell-M} \tag{4a}$$

$$= C(M-1,M-1) \frac{\Gamma(M)^2}{\Gamma(M+\frac{1}{2})\Gamma(M-\frac{1}{2})} , \tag{4b}$$

$$C(0,0) = 1 . \tag{4c}$$

In fact, the special role of the diagonal is due to the existence of a family of commuting diagonal transfer matrices [40,41], both at and also away from the critical point; and it has been exploited in the derivation of the Painlevé III equation of Wu et al. in the scaling limit [9] and of the Painlevé VI equation of Jimbo and Miwa for diagonal correlations at general temperatures [42,43]. Away from the diagonal only a few explicit expressions have been given before at the critical point, i.e. a few short-distance correlations in the isotropic case [4,26,43,44] and the first two terms in the asymptotic expansion of the row correlation function [8,27].

It has been shown recently [16] that the critical spin-spin correlation function also satisfies Hirota's discrete-(imaginary)-time Toda lattice equation [21] with a point source added. More precisely, five "nearby" correlations are related by the quadratic partial difference equation

$$\sinh(2H_c)\,\{C(M,N+1)C(M,N-1) - C(M,N)^2\}$$
$$+ \sinh(2V_c)\,\{C(M+1,N)C(M-1,N) - C(M,N)^2\}$$
$$= 0 \;,\; (M,N) \neq (0,0) \;, \tag{5a}$$
$$C(1,0) = \cosh(2H_c) - \sinh(2H_c)C(0,1) . \tag{5b}$$

We shall see that this equation is extremely powerful when evaluating the correlations.

First we consider the isotropic case, $H_c = V_c$ for which

$$\sinh(2H_c) = \sinh(2V_c) = 1 ,$$

$$C(M,N) = C(N,M) = C(-M,N) . \tag{6}$$

Then all correlations are uniquely determined by (4)-(6). Indeed, if we set M=N in eq. (5a) we obtain

$$C(M,M+1)C(M-1,M) = C(M,M)^2 , \tag{7a}$$
$$C(-1,0) = C(0,1) = 1/\sqrt{2} . \tag{7b}$$

Hence, the ratio

$$\alpha_M \equiv \frac{C(M-1,M)}{C(M,M)} \tag{8}$$

satisifies the simple recursion relation, cf. (4b),

$$\alpha_M \alpha_{M+1} = \frac{\Gamma(M+\frac{1}{2})\Gamma(M+\frac{3}{2})}{\Gamma(M+1)^2} , \tag{9a}$$

with the initial condition, cf. (4c), (7b),

$$\alpha_0 = 1/\sqrt{2} ; \tag{9b}$$

and the solution is

$$\alpha_M = \frac{C(M-1,M)}{C(M,M)} = \frac{1}{\sqrt{2}} \frac{\Gamma(\frac{1}{2}M)\Gamma(M+\frac{1}{2})}{\Gamma(\frac{1}{2}M+\frac{1}{2})\Gamma(M)} . \tag{10}$$

Likewise, setting N = M+1, we obtain from (5a) a linear difference equation for the ratio

$$\beta_M \equiv \frac{C(M-1,M+1)}{C(M,M)} , \tag{11}$$

i.e.

$$\beta_M + \beta_{M+1} = 2\,\alpha_{M+1}^2 \cdot \frac{C(M+1,M+1)}{C(M,M)} , \tag{12a}$$

with the initial condition

$$\beta_0 = \frac{C(-1,1)}{C(0,0)} = \frac{C(1,1)}{C(0,0)} = \frac{\pi}{2} . \tag{12b}$$

This leads to

$$\beta_M = \sum_{\ell=0}^{M-1} (-1)^{M-1-\ell}(\ell+\tfrac{1}{2}) \frac{\Gamma(\frac{1}{2}\ell+\frac{1}{2})^2}{\Gamma(\frac{1}{2}\ell+1)^2} + (-1)^{M-1}\frac{\pi}{2} . \tag{13}$$

The sum can be carried out exactly, splitting the summand using

$$\ell + \frac{1}{2} = 2\left[\left(\frac{1}{2}\ell+\frac{1}{2}\right)^2 - \left(\frac{1}{2}\ell\right)^2\right] . \tag{14}$$

After all cancellations, we have, for M=0, 1, 2,...,

$$\beta_M = \frac{C(M-1,M+1)}{C(M,M)} = 2\left[\frac{\Gamma(\frac{M+2}{2})^2}{\Gamma(\frac{M+1}{2})^2} - \frac{\Gamma(\frac{M+1}{2})^2}{\Gamma(\frac{M}{2})^2}\right] . \tag{15}$$

Continuing this procedure systematically, spin correlations between spins at the origin and in successive rows parallel to the diagonal can be obtained from linear difference equations. The first few results are, for $M \geq 0$,

$$\frac{C(M-1,M+2)}{C(M,M+1)} = 8\frac{\Gamma(\frac{M+2}{2})^4}{\Gamma(\frac{M+1}{2})^4} - 2M(M+1) , \tag{16}$$

$$\frac{C(M-2,M+2)}{C(M,M)} = \frac{16}{3}(M+1)^2 \frac{\Gamma(\frac{M+1}{2})^4}{\Gamma(\frac{M}{2})^4}$$

$$+ \frac{16}{3}(M-1)^2 \frac{\Gamma(\frac{M+2}{2})^4}{\Gamma(\frac{M+1}{2})^4} - \frac{4}{3}M^2(2M^2-1) , \tag{17}$$

$$\frac{C(M-2,M+3)}{C(M,M+1)} = \frac{64}{9}(M+1)^4 \frac{\Gamma(\frac{M+1}{2})^4}{\Gamma(\frac{M}{2})^4} + \frac{32}{9}(M-1)(M+2)(2M^2+2M-1)\frac{\Gamma(\frac{M+2}{2})^4}{\Gamma(\frac{M+1}{2})^4}$$

$$- \frac{8}{9} M^2(M+1)^2(4M^2+4M-5) , \tag{18}$$

etc. Results given in refs. 4,26,43,44 for the first few short-distance correlations at criticality are special cases, with $M = 1,2,3,4$.

3. Anisotropic case

For the nonsymmetric case, ($H_c \neq V_c$), the solution of the Hirota difference equation cannot be determined by the diagonal correlations only. If, however, the spin correlation next to the diagonal $C(M,M+1)$ is also given, then the solution to this discrete Toda-lattice equation is again uniquely determined, much like the linear second-order elliptic difference equation with Cauchy boundary conditions/45/. We have used the transfer matrix mathod/1,2,4,18/ to express $C(M,M+1)$ as an $(M+1)\times(M+1)$ determinant, whose columns, except the last one, are identically the same as that of the diagonal correlation function $C(M+1,M+1)$. At the critical temperature $T=T_c$, the latter correlation function is a Cauchy determinant, of which the cofactors are known in terms of gamma functions/8/. Therefore, we can write $C(M,M+1)$ as a hypergeometric series

$$C(M,M+1) = C(M+1,M+1)\cosh(2H_c)F(\tfrac{1}{2}, M+1; M+\tfrac{3}{2}; -\sinh^2(2H_c)). \tag{19}$$

This is in turn expressible as a Legendre function of the second kind/46/

$$\frac{C(M,M+1)}{C(M+1,M+1)} = \frac{\cosh(2H_c)}{[i\sinh(2H_c)]^{M+1}} \frac{\Gamma(M+\frac{3}{2})}{\Gamma(\frac{1}{2})\Gamma(M+1)} Q_M(z) , \tag{20}$$

with the imaginary argument

$$z \equiv \frac{1-\sinh^2(2H_c)}{2i\sinh(2H_c)} . \tag{21}$$

We note that similar expressions were given by Hartwig for the dimer problem/34/ and we shall come back to this later.

The spin correlations for all other rows parallel to the diagonal can now be calculated using the Hirota difference equation (5a). Setting $N = M+1$ and using (11) for the ratio of correlations, we find the linear difference equation

$$S^{-2}\beta_M + \beta_{M+1} = S^{-2}C^2 \frac{C(M,M+1)^2}{C(M,M)C(M+1,M+1)} , \tag{22}$$

in which

$$S \equiv \sinh(2H_c) = 1/\sinh(2V_c) ; \quad C \equiv \cosh(2H_c) . \tag{23}$$

The right-hand side of (22) is given by (4b) and (20), so we find

$$\beta_{M+1} = \frac{(-1)^{M+1}}{S^{2M+2}} [\beta_0 + \frac{C^4}{2\pi S^2} \sum_{m=0}^{M} (2m+1)Q_m(z)^2] . \tag{24}$$

We now recall the differential equation and recursion relations of the Legendre function/46/

$$(1-z^2)Q_m''(z) - 2zQ_m'(z) + m(m+1)Q_m(z) = 0 , \tag{25}$$

$$(2m+1)zQ_m(z) = (m+1)Q_{m+1}(z) + mQ_{m-1}(z) , \tag{26}$$

$$(1-z^2)Q_m'(z) = (m+1)[zQ_m(z) - Q_{m+1}(z)] , \tag{27a}$$

$$= m[Q_{m-1}(z) - zQ_m(z)] \tag{27b}$$

where the primes denote derivatives with respect to z and all indices have to be nonnegative. One can easily verify from (26) that

$$(t-z)\sum_{m=0}^{M}(2m+1)Q_m(z)^2 = (M+1)[Q_{M+1}(t)Q_M(z) - Q_M(t)Q_{M+1}(z)] - [Q_0(t) - Q_0(z)] . \tag{28}$$

Taking the derivative with respect to t and setting $t=z$, we find

$$\sum_{m=0}^{M}(2m+1)Q_m(z)^2 = (M+1)[Q_M(z)Q_{M+1}'(z) - Q_{M+1}(z)Q_M'(z)] - Q_0'(z) . \tag{29}$$

Using (12b) for β_0 and the identity

$$Q_0'(z) = (1-z^2)^{-1} = 4S^2/C^4 , \tag{30}$$

we find that the correlation function with one spin at the origin and one in the second row parallel to the diagonal is given in terms of a 2×2 Wronskian of Legendre functions, i.e.

$$\beta_M = \frac{C(M-1,M+1)}{C(M,M)} = \frac{(-1)^M MC^4}{2\pi S^{2M+2}} \begin{vmatrix} Q_{M-1}(z) & Q_M(z) \\ Q_{M-1}'(z) & Q_M'(z) \end{vmatrix} , \tag{31}$$

which agrees with (15) for $S=1$.

Now, let us consider the 3×3 Wronskian

$$W(Q_{m-2},Q_{m-1},Q_m) = \begin{vmatrix} Q_{m-2}(z) & Q_{m-1}(z) & Q_m(z) \\ Q_{m-2}'(z) & Q_{m-1}'(z) & Q_m'(z) \\ Q_{m-2}''(z) & Q_{m-1}''(z) & Q_m''(z) \end{vmatrix} . \tag{32}$$

We may eliminate Q_{m-2} using (26) and then eliminate the remaining two second derivatives by (25). We then have

$$W(Q_{m-2},Q_{m-1},Q_m) = \frac{2(2m-1)}{(m-1)(1-z^2)} \{[(1-z^2)Q_{m-1}' - zQ_{m-1}]W(Q_{m-1}, Q_m) + mQ_{m-1}^2Q_m\} . \tag{33}$$

Similarly we may eliminate Q_{m+1} and the second derivatives in $W(Q_{m-1},Q_m,Q_{m+1})$, and then combine the result with (33). This gives

$$(\frac{m+1}{2m+1})W(Q_{m-1},Q_m,Q_{m+1})Q_{m-1}(z) - (\frac{m-1}{2m-1})W(Q_{m-2},Q_{m-1},Q_m)Q_m(z) = 2W(Q_{m-1},Q_m)^2 \quad . \tag{34}$$

Therefore, the ratio

$$\gamma_m = \frac{C(m-2,m+1)}{C(m,m)} = -\frac{m\Gamma(m-\frac{1}{2})C^9}{16\pi^{3/2}\Gamma(m-1)(iS)^{3m+3}} \; W(Q_{m-2},Q_{m-1},Q_m) \tag{35}$$

is the solution of the linear difference equation obtained from the discrete Toda equation (5a) for $M = m-1$, $N = m+1$, cf. (20), (30), (31), (34). Continuing this line of argument, one can show

$$(\frac{m+2}{2m+3})W(Q_{m-1},Q_m,Q_{m+1},Q_{m+2})W(Q_{m-1},Q_m) - (\frac{m-1}{2m-1})W(Q_{m-2},Q_{m-1},Q_m,Q_{m+1})W(Q_m,Q_{m+1})$$

$$= 3W(Q_{m-1},Q_m,Q_{m+1})^2 \; , \tag{36}$$

and $C(m-2,m+2)$ is a 4×4 Wronskian.

Hence, we are led to make the conjecture that the spin correlations at T_c are Wronskians, i.e.

$$\frac{C(M,M+p)}{C(M,M)} = \frac{(C/S)^{p^2}}{2^{\frac{1}{2}p(p-1)}\, i^{p(M+1)} S^{pM}} \prod_{j=1}^{p} \frac{\Gamma(M+j)}{\Gamma(\frac{1}{2})\Gamma(j)\Gamma(M+j-\frac{1}{2})}$$

$$\times \text{ Wronskian } (\{Q_{M+j-1}(z)\}_{j=1}^{p}) \; , \tag{37}$$

where z is defined in (21), S and C in (23).

To summarize the above results, we find that for the anisotropic Ising lattice at the critical point the diagonal correlations are given in terms of gamma functions, see (4); the correlations next-to-the-diagonal are Legendre functions of the second kind; correlations the second row away from the diagonal are 2×2 Wronskians of Legendre functions, etc. As one of the two spins moves a distance p away from the diagonal through the other spin, the correlations are $p \times p$ Wronskians. We have checked (37) by explicit calculations for $p \leq 5$, although a proof for arbitrary p is missing. Setting $S=1$, we reproduce the results of the isotropic case.

4. Asymptotic expansion for large distance

We shall now show how the difference equation (5a) provides an easy way to calculate the successive terms in the asymptotic series for large M and N. There are several smoothness properties that can be derived for $C(M,N)$. (For example, if one writes the transfer matrix as a single exponential, then the long-range contributions in the exponent can be calculated. They decay rapidly enough so the technique[47] developed for the XY-model can be applied. Thus we can analytically continue $C(M,N)$ in M or N, by taking noninteger powers of the transfer matrix, to an entire function of order less than or equal to two, corresponding to decay no faster than Gaussian.) Therefore, when the distance between the two spins is much larger than the lattice spacing, we may use the Taylor expansion

$$\ell nC(M\pm1,N) = \ell nC(M,N) \pm \frac{\partial}{\partial M}\ell nC(M,N) + \frac{1}{2}\frac{\partial^2}{\partial M^2}\ell nC(M,N) + \ldots \quad . \tag{38}$$

Substituting this into (5a) and ignoring terms of order higher than two in the derivatives, we find that the Hirota difference equation becomes

$$\sinh(2H_c)\frac{\partial^2}{\partial N^2}\ell nC(M,N) + \sinh(2V_c)\frac{\partial^2}{\partial M^2}\ell nC(M,N) = 0 \; . \tag{39}$$

At T_c, the vertical and horizontal correlation lengths, ξ_v and ξ_h, are both infinite, but their ratio is

$$\frac{\xi_h}{\xi_v} = \{\frac{\sinh(2H_c)}{\sinh(2V_c)}\}^{1/2} = \sinh(2H_c) = \tan\alpha \; . \tag{40}$$

We scale the coordinates accordingly,

$$N = x/\cos\alpha = R\cos\theta/\cos\alpha \; ,$$

$$M = y/\sin\alpha = R\sin\theta/\sin\alpha \; , \tag{41}$$

and arrive at the Laplace partial differential equation

$$(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})\ell nC(x,y) = 0 \; , \tag{42}$$

(where the order of the arguments of C is changed in taking the continuum limit, reversing an unfortunate but generally accepted convention for the lattice coordinates M,N).

Therefore, in the continuum limit, the difference equation (5a) becomes the rotationally invariant (conformally invariant) Laplace equation (42). A general solution of this is a linear combination of 1, θ, ℓnR, $\theta\ell nR$, $R^{-m}\cos(m\theta)$, and $R^{-m}\sin(m\theta)$. But we have to impose the correct smoothness, boundedness, and reflection-symmetry properties on $C(M,N)$. So the only acceptable solution in the continuum limit is

$$C(x,y) \sim AR^{-\eta} \; , \tag{43}$$

where the constant A and the exponent η are determined by the leading term of the asymptotic expansion for the diagonal correlation[8,44],

$$\ell nC(M,M) \simeq \ell n\frac{A}{M^{1/4}} + \sum_{k=2}^{\infty}\frac{(2^{2k}-1)B_{2k}}{k(k-1)2^{2k}M^{2k-2}} \; , \tag{44}$$

in which[8,48]

$$A = 2^{1/12}e^{3\zeta'(-1)} = 0.64500244850958 \; , \tag{45}$$

$\eta = \frac{1}{4}$, and the B_{2k} are Bernoulli numbers.

To obtain higher order terms in the asymptotic expansion, we write

$$\ell nC(x,y) = \phi_0(R) + \phi_1(\theta,R) + \ldots + \phi_n(\theta,R) + \ldots \; , \tag{46}$$

where the leading term is already given as the solution (43) of the homogeneous Laplace equation (42), i.e.

$$\phi_0(R) = \ln A - \frac{1}{4} \ln R \, . \tag{47}$$

For the higher order terms we have to expect $\phi_n \sim R^{-\sigma_n}$, with $\sigma_n < \sigma_m$ for $n < m$. In fact we can look at the higher terms in (44) and in the asymptotic expansions of (10)[44] and (19). Since (4) and (19) determine all correlations in view of (5a), we have to conclude $\sigma_n = 2n$, $n = 1,2,\ldots$. We note that from ref. 9 we should also expect that the contribution of order $1/R$ vanishes at T_c.

We now can solve for the contributions ϕ_n in (46) systematically by substituting (46) in the Hirota equation (5a) and Taylor expanding as in (38). Collecting all terms of the same order, we find that the ϕ_m are solutions of inhomogeneous Laplace equations where the right-hand sides are expressed in terms of the ϕ_n with $n < m$; the remaining ambiguities can be resolved by either demanding consistency in higher order or, more easily, extracting the integration constants from (44). Since the algebra is straightforward, we present the results without further details:

$$\phi_1(\theta,R) = \frac{1}{2^8 R^2} \{-1-6u \cos(2\theta) + 3\cos(4\theta)\} \, , \tag{48}$$

$$\phi_2(\theta,R) = \frac{1}{2^{13} R^4} \{5+18u \cos(2\theta) + (36+72u^2)\cos(4\theta) - 162u \cos(6\theta) + 63 \cos(8\theta)\} \, , \tag{49}$$

$$\begin{aligned}\phi_3(\theta,R) = \frac{1}{3\cdot 2^{19} R^6} \{&-(524 + 486u^2) - 1566u \cos(2\theta) - (324 + 3672u^2)\cos(4\theta) \\ &- (24003u + 15072u^3)\cos(6\theta) \\ &+ (24732 + 83358u^2) \cos(8\theta) \\ &- 95679u \cos(10\theta) + 28884 \cos(12\theta)\} \, ,\end{aligned} \tag{50}$$

$$\begin{aligned}\phi_4(\theta,R) = \frac{1}{2^{24} R^8} \{&(4307 + 6156u^2) + (24228u + 11016u^3)\cos(2\theta) \\ &+ (-1944 + 28926u^2)\cos(4\theta) + (30312u + 85716u^3)\cos(6\theta) \\ &+ (166428 + 1162836u^2 + 350592u^4)\cos(8\theta) \\ &- (3343968u + 3671004u^3)\cos(10\theta) \\ &+ (1717848 + 8453538u^2)\cos(12\theta) \\ &- 6570156u \cos(14\theta) + 1608657 \cos(16\theta)\} \, ,\end{aligned} \tag{51}$$

$$\begin{aligned}\phi_5(\theta,R) = \frac{1}{5\cdot 2^{28} R^{10}} \{&-(635266 + 2017170u^2 + 468180u^4) - (1985535u + 1501335u^3)\cos(2\theta) \\ &- (45630 + 7129170u^2 + 1928610u^4)\cos(4\theta) \\ &+ (1677195u - 6991965u^3)\cos(6\theta) \\ &- (2262600 + 21305700u^2 + 25366860u^4)\cos(8\theta) \\ &- (253383615u + 585209655u^3 + 103113216u^5)\cos(10\theta) \\ &+ (314470485 + 3263316210u^2 + 1762468290u^4)\cos(12\theta) \\ &- (4196534445u + 6810846165u^3)\cos(14\theta) \\ &+ (1486105290 + 9745245270u^2)\cos(16\theta) \\ &- 5743982880u \cos(18\theta) + 1180103913 \cos(20\theta)\} \, ,\end{aligned} \tag{52}$$

where

$$u \equiv \cos(2\alpha) = \frac{1 - \sinh^2(2H_c)}{1 + \sinh^2(2H_c)} \, . \tag{53}$$

For the correlation along a horizontal row $\theta = 0$, Wu[8,27] has given $\phi_1(0,R) = (1-3u)/128$, in agreement with (48). In the time-continuum limit $u = 1$ (after Wick rotation) our results agree with those for the one-dimensional Ising chain in critical transverse field in the space-like regime, see ref. 38 for $n = 1,2$, all θ, and ref. 39 for $n=3,4,5$, $\theta = \pi/2$.

Finally, the asymptotic series expansion (46)-(53) agrees astonishingly well with the exact results for short distances as implied by our earlier results. Therefore, we plan to use our results to calculate the wavevector-dependent susceptibility. For this, results exist in the scaling limit (near $k = 0$)[26], also for $T \neq T_c$[10]. Recently, Müller and Shrock[39] obtained results for the one-dimensional limit $u = 1$.

5. Dimer problem, Ising model in field $\frac{1}{2} i\pi kT$, and frustrated model

After we had derived our result (19) for the next-to-the-diagonal correlation function, we noted the striking similarity with a result obtained by Hartwig[34] for the monomer-monomer correlation[33] in an otherwise closest-packed dimer problem on a rectangular lattice. In this problem the bonds of the lattice are covered with dimers in such a way that no two dimers meet in any single site. Two given sites (0,0) and (p,q) are left free, that is, they are covered by monomers. All other sites belong to precisely one dimer each. For horizontal dimers we have the weight (activity) x, for vertical dimers y. The monomer-monomer correlation $\omega(p,q)$ is then the ratio of the partition function for all such dimer configurations with two monomers and the partition function of the pure dimer problem without monomers. On the basis of the results of refs. 33 and 34, we then conjectured an identity relating $\omega(p,q)$ with the product of two critical Ising correlations, in the process correcting a few minor errors in ref. 34. We were then able to prove and generalize that identity. It is surprising that this decoupling appears to be new, especially since it was known that the dimer problem is a special case of the eight-vertex model and this eight-vertex model has a decoupling limit in terms of two Ising models.

Our result for the monomer-monomer correlation is

$$\omega(p,q) = \frac{1}{2}(x^2+y^2)^{-1/2}\, C([\tfrac{p}{2}],[\tfrac{q}{2}])\, C([\tfrac{p+1}{2}],[\tfrac{q+1}{2}]),\ \text{for } p+q \text{ odd},$$

$$= 0\ ,\ \text{for } p+q \text{ even}, \tag{54}$$

where [...] denotes integer part, $y/x = \sinh(2H_c)$, and C(M,N) is critical Ising correlation calculated before. We have similar results for multi-monomer correlations. So at one hand the Ising model is a dimer problem on a more complicated lattice[8], on the other hand, the dimer model on a square lattice is two decoupled Ising models!

More generally, since the full monomer-dimer problem is the two-dimensional Ising model in a magnetic field[49], we also looked at the special case of a magnetic field equal to $i\pi kT/2$[6], which reduces to the dimer model for infinite temperature. For this model, we were able to factorize the partition function and all n-point correlations involving order and disorder variables, in terms of two identical Ising models, not necessarily at T_c. As a consequence we could reproduce the existing results[6,35] without difficulty. A special case of such factorization has been noticed before by Forgacs[36], who studied the fully-frustrated Ising model on a square lattice, (see also ref. 37), which is the dual of the model with magnetic field $i\pi kT/2$. He factorized two-spin correlations with spins on the same sublattice in terms of two dual Ising models, using decimation techniques. For the other correlations, our expressions are more complicated, but we believe our results to be useful in a further study of the Yang-Lee edge singularity at high temperatures[50].

6. Further comments, in particular on the role of boundary conditions

We have seen how nicely the Hirota equation (5), with Cauchy boundary conditions (4) and (19), specifies all critical spin-spin correlation functions. But there are many other completely different sets of equations that also specify the correlations. Our conjectured result (37) can be rewritten, using (25)-(27), as a p-th degree homogeneous polynomial in only two consecutive Legendre functions, without derivatives; this polynomial has (at most) p + 1 terms with z-dependent coefficients. Then from the recursion relation (26) it immediately follows that there exists a linear recursion relation, relating the C(M,M+p) for p+2 consecutive values of M. Also for given p and M, there exists a linear differential equation of order p+1 in z determining C(M,M+p). We note that, for $T \neq T_c$, Jimbo and Miwa[42] have found nonlinear recursion relations and a Painlevé VI differential equation for the diagonal correlations only. It is clear that there is a lot more interesting algebraic structure to be discovered.

When we solved for the asymptotic expansion, we effectively used Dirichlet boundary conditions. For the "elliptic" Hirota equation, this is more natural than Cauchy boundary conditions, just as for linear elliptic partial difference equations[45]. The point-source violation (5b) is needed, just as in the linear case, to allow a nontrivial result. Full Dirichlet boundary conditions would involve C(0,0) = 1 and the decay (47) up to o(1) at infinity; but changing the condition at

the origin would only give exponentially small terms at infinity, without affecting the algebraic terms of the asymptotic expansion. We note, however, that in the hyperbolic case the situation is totally different[45]. In that case the Cauchy boundary value problem is stable - not too sensitive to small changes in the initial data; whereas the Dirichlet boundary value problem is extremely unstable, with a light-cone phenomenon and essential differences between space-like and time-like regimes. For the transverse Ising chain, therefore, it was necessary to derive and make use of a Painlevé V differential equation in order to determine the full asymptotic behaviour[38]. For the present case a Painlevé type difference equation should exist determining correlations in rows, but this is not crucial, although it may still be useful to have it.

Finally, several constructive existence and uniqueness proofs can be given for the Hirota equation for $T = T_c$, adopting methods for the linear case[45]. We note that for $T \neq T_c$ McCoy and Wu[51] have proved results of this type for the isotropic case, studying the lattice Painlevé equation for $T \neq T_c$. In particular they proved a maximum-modulus theorem. Their methods can be extended to the anisotropic case; but extension to the present case requires extra care since their method makes a hidden use of the existence of a mass, i.e. $T \neq T_c$.

In order to give the existence and uniqueness proofs we assume a (finite) set of interior points S and a set of boundary points ∂S. Then on ∂S we assume $\phi(M,N) \equiv \ln C(M,N)$ to be given and uniformly bounded, i.e. $0 \geq \phi(M,N) \geq \ln\varepsilon$, with ε a small but positive number. The rest of the Dirichlet boundary value problem is that on S we have (5a), or

$$\phi = \frac{\alpha+\beta+\gamma+\delta}{4} + \frac{1}{2}\ln\{\cosh(\tfrac{\psi+\alpha+\beta-\gamma-\delta}{2})/\cosh(\tfrac{\psi}{2})\}\ , \tag{55}$$

where

$$\phi \equiv \phi(M,N)\ ,\ \alpha \equiv \phi(M+1,N)\ ,\ \beta \equiv \phi(M-1,N),$$

$$\gamma \equiv \phi(M,N+1)\ ,\ \delta \equiv \phi(M,N-1)\ ,\ \psi \equiv \ln\{\sinh(2V_c)/\sinh(2H_c)\}\ . \tag{56}$$

The existence proof follows, iterating eq. (55) for internal points, starting with $\phi^{(0)}(M,N) \equiv \ln\varepsilon$ on S. For each point of S we then have a bounded increasing sequence, which converges. The uniqueness proof is more subtle. First we notice from (55) that if α,β,γ, and δ are allowed to change only by an amount between $-\Delta$ and Δ, that φ can also only change by an amount in this range. Then we assume to have two solutions ϕ_1 and ϕ_2, equal on the boundary ∂S, but differing by at most Δ, i.e. $|\phi_1-\phi_2| \leq \Delta$. Now there are internal points with a horizontal and a vertical boundary point; for such points we can then prove $|\phi_1-\phi_2| \leq \frac{1}{2}\Delta$. Next there are points bordering to a boundary point and a $\frac{1}{2}\Delta$ point. For such a point $|\phi_1-\phi_2| \leq \frac{3}{4}\Delta$. Continuing through all, say L, points we arrive at the conclusion $|\phi_1-\phi_2| \leq (1-2^{-L})\Delta$, and $\Delta = \max_S|\phi_1-\phi_2| \leq (1-2^{-L})\Delta$. Hence Δ=0, and uniqueness is proved. We note that if we would have started with an infinite interior set S we would have needed some asymptotic form, so we can effectively compare with a finite set up to some error which decreases while letting this finite set increase.

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