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NEW RESULTS FOR SUSCEPTIBILITIES IN PLANAR ISING MODELS*

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We briefly review recent progress on calculating susceptibilities in planar Ising models.

1. Introduction

First of all, a project like the current one cannot be undertaken by a single person. We owe a lot to our collaborators, teachers, and colleagues, especially R.J. Baxter, H.W. Capel, A.J. Guttmann, M. Jimbo, B.-Q. Jin, X.-P. Kong, T. Miwa, B.M. McCoy, B.G. Nickel, W.P. Orrick, M. Sato, and T.T. Wu. The literature on the two-dimensional Ising model also is very extended. Therefore, we shall only give limited citations, and encourage the interested reader to consult the quotations in these references. Most of the current work is a brief review of results in Refs. 1–4.

The symmetric two-dimensional Ising model is defined by

$$\mathcal{H} = -J\sum_{m,n}(\sigma_{m,n}\sigma_{m,n+1}+\sigma_{m,n}\sigma_{m+1,n}). \tag{1}$$

For this model it is convenient to define elliptic modulus[5]

$$k = 1/\sinh^2(2J/k_{\rm B}T), \tag{2}$$

which is <1 for $T<T_{\rm c}$ and >1 for $T>T_{\rm c}$, with $k\to 1/k$ giving the Kramers-Wannier duality transformation.

The spontaneous magnetization is simply given by[6,7]

$$\langle\sigma\rangle = \begin{cases} (1-k^2)^{1/8}, & T<T_{\rm c}, \\ 0, & T\ge T_{\rm c}. \end{cases} \tag{3}$$

The calculation of the pair correlation function

$$C(m,n) = \langle\sigma_{0,0}\sigma_{m,n}\rangle \tag{4}$$

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is more involved and can be carried out using quadratic difference equations[8]

$$\begin{aligned}
&\big[C(m,n{+}1)C(m,n{-}1) - C(m,n)^2\big] \\
&\qquad +k\left[C^*(m{+}1,n)C^*(m{-}1,n) - C^*(m,n)^2\right] = 0, \qquad (5)\\
&\big[C(m+1,n)C(m{-}1,n) - C(m,n)^2\big] \\
&\qquad +k\left[C^*(m,n{+}1)C^*(m,n{-}1) - C^*(m,n)^2\right] = 0, \qquad (6)
\end{aligned}$$

where $C^*(m,n)$ is the dual correlation function obtained by replacing $k \to 1/k$. For the symmetric case (1), these two equations are equivalent. To solve them we need initial conditions. For $T = T_{\rm c}$, we have

$$C(n,n) = C^*(n,n) = \prod_{j=1}^{n} \frac{\Gamma(j)^2}{\Gamma(j+\frac{1}{2})\Gamma(j-\frac{1}{2})}. \qquad (7)$$

which form was already known to Onsager and Kaufman.[9] For $T \neq T_{\rm c}$, $C(n,n)$ and $C^*(n,n)$ can be calculated by Toeplitz determinants[1,4,7]

$$C(n,n) = (-1)^n \det_{1\le i,j\le n} \big(\{a_{i-j-1}\}\big), \qquad (8)$$

$$C^*(n,n) = \det_{1\le i,j\le n} \big(\{a_{i-j}\}\big), \qquad (9)$$

where

$$a_n = (2nk^{-1}a_{n-1} + a_{-n})/(2n+1), \qquad (10)$$

$$a_{-n-1} = (2nka_{-n} + a_{n-1})/(2n+1), \qquad (11)$$

for $n = 1, 2, \cdots$, with the initial conditions

$$a_0 = \frac{2}{\pi k}\big[\mathrm{E}(k) - (1-k^2)\mathrm{K}(k)\big], \quad a_{-1} = -\frac{2}{\pi}\,\mathrm{E}(k). \qquad (12)$$

However, it can be done faster by another set of quadratic difference equations due to Jimbo and Miwa.[10]

2. High- and Low-Temperature Series for Susceptibility

Very recently, with the help of (5) the high- and low-temperature series for the susceptibility were much extended by the authors of Ref. 2. In terms of the reduced susceptibility,

$$\bar\chi \equiv k_{\rm B}T\chi = \sum_{m,n=-\infty}^{\infty} \big(\langle\sigma_{0,0}\sigma_{m,n}\rangle - \langle\sigma_{0,0}\rangle^2\big), \qquad (13)$$

they found for $T > T_{\rm c}$,

$$\begin{aligned}
\bar\chi = {}& 1 + 4s_{\rm h} + 12s_{\rm h}^2 + 32s_{\rm h}^3 + 76s_{\rm h}^4 + 176s_{\rm h}^5 + 400s_{\rm h}^6 + \cdots \\
&+ 2007330258829172991431166572284107035662323251845 3\backslash \\
&\quad 67545550226445723763406738301159160108585998318576\, s_{\rm h}^{323}, \quad (14)
\end{aligned}$$

with $s_{\mathrm{h}} \equiv s/2 \equiv \sinh(2K)/2$, and for $T < T_{\mathrm{c}}$,

$$\begin{aligned}\bar{\chi} = \; & 4s_{\mathrm{l}}^4 + 16s_{\mathrm{l}}^6 + 104s_{\mathrm{l}}^8 + 416s_{\mathrm{l}}^{10} + 2224s_{\mathrm{l}}^{12} + \cdots \\ & + 30515477245090443508556620725003894684638932739 07\backslash \\ & \quad 573281021122943429942084961223451717498203084 5245\backslash \\ & \quad 5331887458424846630637797467206682914215700492366\backslash \\ & \quad 927125970737985527522487370743555011446200114 4064\, s_{\mathrm{l}}^{646}, \end{aligned} \tag{15}$$

with $s_{\mathrm{l}} \equiv 2/s \equiv 2/\sinh(2K)$. The size of the coefficients may look ridiculous at first sight. However, it is well-known to series expanders that the new information in each successive coefficient is often in the last few digits.

Near the ferromagnetic critical point, the susceptibility behaves asymptotically as

$$\frac{\beta^{-1}\chi_{\pm}}{\left(\sqrt{1+\tau^2}+\tau\right)^{1/2}} \approx C_{0_{\pm}} (2K_{\mathrm{c}}\sqrt{2})^{7/4} |\tau|^{-7/4} \hat{F}_{\pm} + \hat{B}_{\mathrm{f}}, \tag{16}$$

where $(\sqrt{1+\tau^2}+\tau)^{1/2} = 1/\sqrt{s}$ and $\tau = (1/s - s)/2$, and $\pm$ stands for T above or below T_{c}. In (16) the ferromagnetic background is given by[2]

$$\begin{aligned}\hat{B}_{\mathrm{f}} = \; & (-0.1041332450938310264521601268604734337162367273 14 \\ & \quad -0.074368869753207080019958591697995003280476320 28\tau \\ & \quad -0.008144713909119599537154285865572389326605774 0\tau^2 \\ & \quad +0.004504107712232015926355020852986970591364528\tau^3 \\ & \quad + \cdots - 0.16279253648974618861881216566686\tau^{14}) \\ & +(\log|\tau|)\times \\ & \quad (0.032352268477309406090656526721221666637730948898\tau \\ & \quad -0.005775529379688463009148756401320101367715298 0\tau^3 \\ & \quad + \cdots - 0.041428586463052869356803144137620\tau^{14}) \\ & +(\log|\tau|)^2\times \\ & \quad (0.009391569871145872131795331872707577064951365 4\tau^4 \\ & \quad -0.008695925462879238021564166451917529879129 22\tau^6 \\ & \quad + \cdots - 0.0055571002151161308034896964314679\tau^{14}) \\ & +(\log|\tau|)^3\times \\ & \quad (-0.000015771569138451840480001012621461738178\tau^9 \\ & \quad +0.000034428206620888755364779985685775338 0\tau^{11} \\ & \quad -0.0000052442717748722617416158377914939 3\tau^{13}), \end{aligned} \tag{17}$$

whereas, the ferromagnetic scaling amplitudes functions are given by[2]

$$\begin{aligned}\hat{F}_{+} & = 1 + \tau^2/2 - \tau^4/12 - 0.1235292285752086663\tau^6 \\ & \quad +0.136610949809095\tau^8 - 0.13043897213\tau^{10} + \cdots, \text{ for } T > T_{\mathrm{c}}, \end{aligned} \tag{18}$$

$$\begin{aligned}\hat{F}_{-} & = 1 + \tau^2/2 - \tau^4/12 - 6.3213068404959366230 67\tau^6 \\ & \quad +6.2519974704602432 9\tau^8 - 5.6896599756180\tau^{10} + \cdots, \text{ for } T < T_{\mathrm{c}}. \end{aligned} \tag{19}$$

More coefficients are given in Ref. 2. The last digit in each term above may not be reliable. As we have normalized[a] $F_{\pm}(\tau) \to 1$ for $T \to T_{\mathrm{c}}$ (or $\tau \to 0$), we need to give also the leading susceptibility amplitudes:

$$\begin{aligned} C_0^+ &= 1.000815260440212647119476363047210\backslash \\ &\qquad 236937534925597789\,(2K_c\sqrt{2})^{-7/4}\sqrt{2}, \\ C_0^- &= 1.000960328725262189480934955172097\backslash \\ &\qquad 320572505951770117\,(2K_c\sqrt{2})^{-7/4}\sqrt{2}/(12\pi). \end{aligned} \tag{20}$$

Near the antiferromagnetic critical point, the susceptibility behaves as

$$\frac{\beta^{-1}\chi}{\left(\sqrt{1+\tau^2}+\tau\right)^{1/2}} \approx \hat{B}_{\mathrm{af}}, \tag{21}$$

where

$$\begin{aligned} \hat{B}_{\mathrm{af}} = \;& 0.1588665229609474882333592313690210116925239008416 \\ & \quad +0.1495668369385359051943820294335912863747112072 62\tau \\ & \quad +0.01071222587983288033470968550659996768542030678\tau^2 \\ & \quad +\cdots+0.007123677682511208149032476379667\tau^{14} \\ & +(\log|\tau|) \\ & \;\;(-0.155317190158011058593413353893273452999212160030 5\tau \\ & \quad +0.03206714814586975221843437287457551882247161782\tau^3 \\ & \quad +\cdots-0.009405623038076560771947492508864 9\tau^{14}) \\ & +(\log|\tau|)^2 \\ & \;\;(0.01153371437882328027949011442761203640684043805\tau^4 \\ & \quad -0.01131173492069156006753505653220784271640568 4\tau^6 \\ & \quad +\cdots-0.006744701894515262884782000593434 32\tau^{14}) \\ & +(\log|\tau|)^3 \\ & \;\;(0.000057899719476487729776006722114406224954 1\tau^9 \\ & \quad -0.000169915088240128902407964467449359088 12\tau^{11} \\ & \quad +0.00032664884687465587957270016883093909\tau^{13}). \end{aligned} \tag{22}$$

The difference of $\hat{F}_+$ and $\hat{F}_-$ in Eqs. (18) and (19) implies that a suggestion of Aharony and Fisher[11] breaks down in higher order. They had brought up the possibility that there are "no irrelevant variables." This they concluded from the speculation that the Ising model free energy in the critical region can be described entirely by two nonlinear but "analytic" (thermal and magnetic) scaling fields. Then the scaling amplitude can be found to be

$$\hat{F}_{\pm} = 1 + \frac{1}{2}\tau^2 - \frac{31}{384}\tau^4 + \frac{125}{3072}\tau^6 + \mathrm{O}(\tau^8), \tag{23}$$

equal above and below T_{c}.[2]

[a]Note that we have a slight change of notation with respect to Ref. 2, as we have rescaled all B's and F's with a factor $\sqrt{s}$.

We now know that this simple picture is incomplete and that corrections to scaling due to breaking of rotational symmetry must be considered. Indeed, the correlation functions have a kind of multipole long-distance expansion,[12] which can explain the deviations from fourth order on. Very recently, a conformal field theory explanation has also been given.[13] To study the effect in more detail we shall have to study the model on other lattices.

Another interesting feature discussed in Ref. 2 is that the susceptibility has a natural boundary at the critical point, i.e. there exists a closed curve of (essential) singularities fully prohibiting analytic continuation in the complex temperature plane from high to low temperatures. The Ising susceptibility is not differentiably finite, unlike the zero-field free energy and the spontaneous magnetization. This then explains why there is no simple closed form expression available after half a century of research. Yet, we now have algorithms of polynomial complexity, which is as good for numerical analysis.

3. Baxter's Z-invariant inhomogeneous Ising model

Baxter's Z-invariant Ising model is defined in terms of a set of oriented straight lines carrying "rapidity" variables u_i, v_j, $\cdots$. In the scaling limit the scaled correlation function depends on a single distance variable R, as first discovered by Bai-Qi Jin,[1]

$$R = \frac{1}{2}\left[\left\{\sum_{j=1}^{2m}\cos(2u_j)\right\}^2 + \left\{\sum_{j=1}^{2m}\sin(2u_j)\right\}^2\right]^{1/2}. \tag{24}$$

This is given in terms of the $2m$ rapidity variables crossing between the two spins in question. Using the diagonal correlation length $\xi_{\rm d}$ to introduce the scaled distance

$$r = R/\xi_{\rm d}, \quad \text{where} \quad \xi_{\rm d}^{-1} = |\log k|, \tag{25}$$

we have found the most general form of the scaled correlation functions to be

$$\langle\sigma\sigma'\rangle \approx |1-k^{-2}|^{1/4}F(r), \quad \langle\sigma\sigma'\rangle^* \approx |1-k^{-2}|^{1/4}G(r), \tag{26}$$

where the functions $F(r)$ and $G(r)$ satisfy

$$FF'' - F'^2 = -r^{-1}GG', \qquad GG'' - G'^2 = -r^{-1}FF', \tag{27}$$

and the front factor is the square of the spontaneous magnetization for $T < T_{\rm c}$ or $k > 1$. $F(r)$ and $G(r)$ are the Painlevé functions for the uniform rectangular Ising lattice,[14] see Refs. 1, 4 for more details.

4. Susceptibility in Z-Invariant Lattice

For a general ferromagnetic Z-invariant lattice with $\mathcal{N}$ sites, the susceptibility χ is given by

$$\bar\chi \equiv k_{\rm B}T\chi = \lim_{\mathcal{N}\to\infty}\frac{1}{\mathcal{N}}\sum_{m_1,n_1}\sum_{m_2,n_2}\big(\langle\sigma_{m_1,n_1}\sigma_{m_2,n_2}\rangle - \langle\sigma_{0,0}\rangle^2\big), \tag{28}$$

where (m_1, n_1) and (m_2, n_2) run through the possible coordinates of the spins. In periodic cases one of the two sums can be done trivially. In quasiperiodic cases this can only be done asymptotically at the largest distance scale. Hence, in the scaling limit and for both periodic and quasiperiodic Z-invariant lattices, $\bar{\chi}$ becomes

$$\bar{\chi} \approx g_0 \int_{-\infty}^{+\infty} \mathrm{d}M \int_{-\infty}^{+\infty} \mathrm{d}N \, \frac{\check{F}_{\pm}(\kappa R)}{R^{1/4}}, \tag{29}$$

where

$$\frac{\check{F}_{-}(\kappa R)}{R^{1/4}} = |1 - k^{-2}|^{1/4} \big(G(R/\xi_{\mathrm{d}}) - 1 \big), \tag{30}$$

$$\frac{\check{F}_{+}(\kappa R)}{R^{1/4}} = |1 - k^{-2}|^{1/4} \, F(R/\xi_{\mathrm{d}}), \tag{31}$$

$\kappa = 1/\xi_{\mathrm{d}} = |\log k|$, and R reduces to

$$R = \sqrt{aM^2 + 2bMN + cN^2} \tag{32}$$

with a, b, and c known constants that can be calculated choosing suitable integer coordinates M and N. Also, g_0 is the corresponding multiplicity factor counting how many spin distance vectors fall exactly or asymptotically within a unit cell in the (M, N) plane. Therefore,[4]

$$\bar{\chi} = \frac{2\pi g_0}{\sqrt{ac - b^2}} \int_0^{\infty} \mathrm{d}r \, r^{3/4} \check{F}_{\pm}(r) \, \kappa^{-7/4} + \cdots = A_{\pm} |t|^{-7/4} + \mathrm{O}(|t|^{-3/4}), \tag{33}$$

with $t \equiv |T - T_{\mathrm{c}}|/T_{\mathrm{c}}$, giving the exact $T > T_{\mathrm{c}}$ and $T < T_{\mathrm{c}}$ susceptibility amplitudes for all periodic and quasiperiodic Z-invariant lattices.

Note that this result implies that the ratio A_+/A_- is universal for all periodic and quasiperiodic ferromagnetic Z-invariant Ising models. This may be the first time that this is shown to this generality for the magnetic susceptibility. For the analysis of the long susceptibility series in the isotropic square lattice A_+ and A_- were evaluated to very high precision by Nickel.

Therefore, we can now give A_+ and A_- for the isotropic square (sq), triangular (tr) and honeycomb (hc) lattices to many places, i.e.[4]

$$\begin{aligned} A_+^{\mathrm{sq}} &= 0.9625817323087721140443298094334694951671391947579365, \\ A_+^{\mathrm{tr}} &= 0.9242069582451643296971575778559317176696261520028389, \\ A_+^{\mathrm{hc}} &= 1.046417076152338359733871672674357433252295746539088, \\ A_-^{\mathrm{sq}} &= 0.02553697452202390538595345622639847192921968727077455, \\ A_-^{\mathrm{tr}} &= 0.02451890447700000489080855239719772023653022851422950, \\ A_-^{\mathrm{hc}} &= 0.02776109842539704507743379795258285503609969877633251. \end{aligned} \tag{34}$$

Also, more generally,

$$\bar{\chi} = \frac{2\pi g_0}{\sqrt{ac - b^2}} \, |\log(k)|^{-7/4} \int_0^{\infty} \mathrm{d}r \, r^{3/4} \check{F}_{\pm}(r) \tag{35}$$

is a product of a factor depending on rapidities and the modulus and a factor which is a universal integral over a Painlevé V function. Hence, the amplitudes are known—in principle—to this high accuracy for all Z-invariant (quasi)-periodic cases. We plan to use these values later to analyze long series for the isotropic triangular and honeycomb lattices, once they are available.

We note that the numbers given above agree to a few places with earlier series extrapolations. Four of the six agree to about ten places with those of Wu et al.[14] and of Vaidya.[15] For T above $T_{\rm c}$, they agree to better than three places with those obtained from the Syozi-Naya[16] approximation, but this can be understood as this approximation is precisely the $\chi^{(1)}_{<}$ approximation in Wu et al.

5. Outlook

We are working to extend and analyze series for other lattices in order to get more information on irrelevant variables in the corrections to scaling, having a preliminary algorithm of polynomial complexity for the isotropic honeycomb and triangular lattices which reproduces the known series coefficients. But more work needs to be done to increase its efficiency, as we will need to go to one to two hundred terms, before being able to see clearly the effect of the irrelevant variables.

We are also looking at the susceptibility of Ising models on Penrose tilings. Finally, we also want to look at the effect of frustration, which occurs in the regime where elliptic modulus k is purely imaginary.

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