Correlation Functions and Susceptibility in the Z-Invariant Ising Model

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ABSTRACT We discuss historical and recent developments in the calculation of two-point correlation functions and susceptibilities in planar Ising models. We note some remarkable simplifications of approach within Baxter's Z-invariant generalization of the usual uniform model. We highlight several of the many important contributions of McCoy to the field.

1 Introduction

It is a great pleasure to write a chapter on the Ising model in honor of the sixtieth birthday of Dr. Barry M. McCoy. Both of us have profited from many years of fruitful collaboration and correspondence with him, most of all on the planar Ising model.

The history of the Ising model goes back more than eighty years to a suggestion by Lenz and the subsequent thesis work of Ising [1] on the special case of a one-dimensional chain. After this modest beginning it was up to others to continue the work. Ising, being Jewish with a non-Jewish wife, was spared the worst of the Holocaust, but he was unable to follow the subsequent developments until much later [2]. In the mean time, there had been many developments culminating in the exact determination of the critical point of the square-lattice Ising model by Kramers and Wannier [3] and the calculation of its free energy by Onsager [4].

Since that time, tens of thousands of papers have appeared, wholly or partially devoted to the Lenz–Ising model, a name preferred by Ising himself [2]. In the current paper, we shall restrict our discussion to the exact or extremely accurate evaluation of the two-point correlation function and

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the magnetic susceptibility of the symmetric two-dimensional Ising model and its Z-invariant extension [5], which covers the rectangular, triangular, honeycomb, kagome, and checkerboard lattices as special cases.

The ferromagnetic symmetric two-dimensional Ising model is defined by the interaction energy

$$\mathcal{H} = -J \sum_{m,n} (\sigma_{m,n}\sigma_{m,n+1} + \sigma_{m,n}\sigma_{m+1,n}), \tag{1.1}$$

where state $\{\sigma\}$ is defined by a map assigning to each site (m,n) of a square lattice a spin $\sigma_{m,n} = \pm 1$ and where J is a positive constant. For this model it is convenient to define elliptic modulus [4]

$$k = \sinh^2(2J/k_{\mathrm{B}}T) \equiv k_>, \tag{1.2}$$

which is < 1 for $T > T_{\mathrm{c}}$ and > 1 for $T < T_{\mathrm{c}}$, with $k \to 1/k \equiv k_<$ giving the Kramers-Wannier duality transformation [3]. In terms of modulus k, the spontaneous magnetization is simply given by [6–8]

$$\langle\sigma\rangle = \begin{cases} (1-k^{-2})^{1/8}, & T < T_{\mathrm{c}}, \\ 0, & T \geq T_{\mathrm{c}}, \end{cases} \tag{1.3}$$

even though its derivation is quite involved. There are three main methods available: the Szegö–Kac theorem for the asymptotic evaluation of Toeplitz determinants [9–13], Fredholm integral equation [7] for the matrix element connecting the odd and even groundstates in the spinor approach [14, 15], and corner transfer matrices [16].

The calculation of the pair correlation function

$$C(m,n) = \langle\sigma_{0,0}\sigma_{m,n}\rangle - \langle\sigma\rangle^2 \tag{1.4}$$

started with the work of Kaufman and Onsager [15], who were the first to calculate it in terms of determinants [9]. Fisher [17–19] used these results and found the critical exponent of the susceptibility to be $\gamma = 7/4$. He also found an exact result for the perpendicular susceptibility of the Ising model [20], which we shall not discuss here. The long-distance asymptotics of the pair correlation function was first properly studied in the mid sixties [21–24] and a detailed treatment is given in the well-known textbook of McCoy and Wu [13]. This book also presents a host of other results of McCoy and Wu, such as the essential singularity in the layered-random Ising model [25], which are outside the scope of the present work.

Next, in 1973, it was announced that the scaled spin-spin correlation function of the Ising model can be calculated solving a nonlinear ordinary differential equation of Painlevé III type [26–31]. This amazing result did not only lead to an accurate computation of the scaled correlation function and the amplitudes of the power-law singularity of the susceptibility above and below T_c. It inspired Sato, Miwa, and Jimbo to generalize the result via a novel connection with deformation theory, from which they could derive a system of nonlinear partial differential equations for the n-spin correlation functions [32]. A detailed discussion of how these results fit in within a wider context of statistical mechanics and quantum field theory has been given a few years ago by McCoy [33].

The original derivations of these nonlinear differential equations started out with perturbation expansions of the correlation functions in terms of infinite series, in which the successive terms are multiple integrals with a forever increasing number of integrations. The convergence of these series has been studied and the integrals have been simplified [34–37]. There is also an equivalent Painlevé V description of the correlation functions [38–40], which we shall discuss in more detail below in Section 8.

Another development in the seventies was the discovery that the infinite-temperature time-dependent correlation functions of the inhomogeneous transverse Ising chain satisfy the Toda differential-difference equation [41]. The question arose next if the scaling-limit continuum partial differential equations for the two-dimensional Ising model correlation functions could be generalized to nonlinear partial difference equations on the lattice. The answer to this turned out to be yes [42–49]. The simplest case is Hirota's discrete generalization of the Toda lattice [44].

Such bilinear equations can be derived for general planar Ising lattices using a generalization of Wick's theorem [43, 48]. For the uniform lattice they can also be derived by substituting the aforementioned perturbation expansions in terms of multiple integrals [46]. This latter derivation is more laborious than the former method, but it leads to another generalization relaxing the expansion parameter $2/\pi$ for Ising to general complex λ. The physical meaning of this generalization is of yet not clear.

In the present work we shall discuss the current status of calculating the two-point correlation function and the susceptibility in regular planar Ising models. We shall also explain how Z-invariance can be used to obtain new results and to generalize existing results through alternative simpler derivations. In Section 2 we shall review the most efficient scheme available to calculate the correlations of the square lattice and in Section 3 we shall discuss how this has been used to obtain results for the susceptibility. In Section 4 we shall introduce Baxter's general Z-invariant Ising model and in Section 5 we shall discuss the equations for the pair-correlation function in this model. Jin's conjecture for the correlation function in the scaling limit is presented in Section 6 and a proof is outlined in Sections 7, 8, 9, together with some further discussion. This scaling-limit result is applied in Section 10 to the leading singularity of the susceptibility in the Z-invariant Ising model. In Section 11 we illustrate how the quadratic relations for correlations in the general planar Ising model, and Z-invariance determine those correlations. In Section 12 we give some conclusions and outlook.

2 Difference Equations for Correlation Functions

As we shall treat the general Z-invariant case later, we can now restrict ourselves to the simplest case of a symmetric uniform square Ising lattice. The quadratic difference equations for the spin-spin correlation function [44] for this case (1.1) reduce to:

$$\begin{aligned}&\big[C(m,n{+}1)C(m,n{-}1)-C(m,n)^2\big]\\&\quad+k\big[C^*(m{+}1,n)C^*(m{-}1,n)-C^*(m,n)^2\big]=0,\end{aligned}\tag{2.1}$$

$$\begin{aligned}&\big[C(m{+}1,n)C(m{-}1,n)-C(m,n)^2\big]\\&\quad+k\big[C^*(m,n{+}1)C^*(m,n{-}1)-C^*(m,n)^2\big]=0,\end{aligned}\tag{2.2}$$

$$\begin{aligned}&\big[C(m,n)C(m{+}1,n{+}1)-C(m{+}1,n)C(m,n{+}1)\big]=\\&k\big[C^*(m,n)C^*(m{+}1,n{+}1)-C^*(m{+}1,n)C(m,n{+}1)\big],\end{aligned}\tag{2.3}$$

$$\begin{aligned}&\sqrt{k}\,\big[C(m{+}1,n)C^*(m{-}1,n)+C(m{-}1,n)C^*(m{+}1,n)\\&\quad+C(m,n{+}1)C^*(m,n{-}1)+C(m,n{-}1)C^*(m,n{+}1)\big]\\&\quad=\ (k{+}1)\,C(m,n)C^*(m,n),\end{aligned}\tag{2.4}$$

with

$$C(0,0)=C^*(0,0)=1,\quad C(1,0)+\sqrt{k}\,C^*(0,1)=\sqrt{k{+}1},\tag{2.5}$$

$$\begin{aligned}C(m,n)&=C(n,m)=C(|m|,|n|),\\C^*(m,n)&=C^*(n,m)=C^*(|m|,|n|).\end{aligned}\tag{2.6}$$

Here $C^*(m,n)$ is the dual correlation function, *i.e.* the correlation function at the dual temperature obtained by replacing $k\to 1/k$ in (1.2). Kadanoff and Ceva [50] would call $C(m,n)$ and $C^*(m,n)$ the correlation functions of order- and disorder variables, respectively. Equations (2.1), (2.2), and (2.4) do not hold if $m=n=0$, in which case they are replaced by (2.5).

These equations form an overdetermined (holonomic [32]) system. In the symmetric case with (2.6), eqs. (2.1) and (2.2) are equivalent and we can calculate all other correlation functions recurrently, once we know how to calculate the diagonal correlation functions $C(n,n)$ and $C^*(n,n)$. Of course, the nearest-neighbor correlation functions $C(0,1)$ and $C^*(0,1)$ are known, since they are proportional to the internal energy [4, 13]. We can solve

$C(n,n+1)$ and $C^*(n,n+1)$ from (2.1) and (2.4) with $m=n=1,2,\cdots$. Then the remaining correlation functions, with $n-m=1,2,\cdots$, can be solved from (2.1) and (2.2).

At the critical temperature $T=T_{\rm c}$ or $k=1$ with $C^*(m,n)\equiv C(m,n)$, we only need (2.1) and

$$C(n,n)=\prod_{j=1}^{n}\Gamma(j)^2/\big(\Gamma(j+\tfrac12)\Gamma(j-\tfrac12)\big), \tag{2.7}$$

which was essentially already known to Onsager and Kaufman [13, 15].

When $T\neq T_{\rm c}$, we can use the well-known Toeplitz determinants [13,51]. It is, however, more efficient to use the equations provided by Jimbo and Miwa [38]. Denoting $C^*(n,n)\equiv A_n$ and $C(n,n)\equiv C_n$, these can be rewritten as:

$$B_{n+1}=-(kA_n^+B_n^+ + k^{-1}A_n^-B_n^-)/\big((2n+3)A_n\big), \tag{2.8}$$

$$C_{n+1}^{\pm}=(A_{n+1}C_n^{\pm}-C_n\,A_n^{\pm})/(k^{\pm1}A_n), \tag{2.9}$$

$$D_{n+1}^{\pm}=(A_{n+1}D_n^{\pm}+C_n\,B_n^{\pm})/A_n\,, \tag{2.10}$$

$$C_{n+1}=-(C_{n+1}^{+}D_{n+1}^{+}+C_{n+1}^{-}D_{n+1}^{-})/\big((2n+1)A_n\big), \tag{2.11}$$

$$A_{n+1}^{\pm}=(A_{n+1}A_n^{\pm}-B_{n+1}C_{n+1}^{\pm})/A_n\,, \tag{2.12}$$

$$B_{n+1}^{\pm}=(k^{\pm1}A_{n+1}B_n^{\pm}+B_{n+1}D_{n+1}^{\pm})/A_n\,, \tag{2.13}$$

$$A_{n+2}=(A_{n+1}^2-B_{n+1}C_{n+1})/A_n\,, \tag{2.14}$$

which can be solved iteratively, in the above order for $n=0,1,\cdots$, starting from the initial conditions

$$\begin{aligned}
&A_0=B_0=C_0=1,\quad A_1=2\mathrm{E}(k)/\pi,\\
&B_0^+=D_0^-=k',\quad C_0^+=A_0^-=1/k',\quad D_0^+=C_0^-=0,\\
&A_0^+=2\big(2\,\mathrm{E}(k)-\mathrm{K}(k)\big)/(\pi k'),\quad B_0^-=2k'\big(\mathrm{K}(k)-\mathrm{E}(k)\big)/\pi,
\end{aligned} \tag{2.15}$$

where $\mathrm{K}(k)$ and $\mathrm{E}(k)$ are the usual complete elliptic integrals, $k'=\sqrt{1-k^2}$.

Remarks.

Jimbo and Miwa [38] discovered the above system (2.8)–(2.15) through deformation theory. A generalization of their result to the row or the next-to-the-diagonal correlation functions would be very useful. We believe that there must exist a general underlying theorem on Toeplitz determinants whose entries satisfy simple recurrence relations. At the very least, there should certainly exist a more transparent and direct derivation starting from the Toeplitz representations of $C(n,n)$ and $C^*(n,n)$ and the linear recurrence relations of their entries [51]*

$$\begin{aligned} a_n &= (2nk^{-1}a_{n-1} + a_{-n})/(2n+1), \\ a_{-n-1} &= (2nka_{-n} + a_{n-1})/(2n+1), \quad n = 1, 2, \cdots, \end{aligned} \tag{2.16}$$

with the initial conditions

$$a_0 = \frac{2}{\pi k}\left[\mathrm{E}(k) - (1-k^2)\mathrm{K}(k)\right], \quad a_{-1} = -\frac{2}{\pi}\,\mathrm{E}(k). \tag{2.17}$$

Alternatively, (2.16) can be replaced by

$$(2n+3)a_{n+1} - 2\big((n+1)k^{-1} + nk\big)a_n + (2n-1)a_{n-1} = 0, \tag{2.18}$$

iterating with this equation in the positive and negative n directions.

It is well known [9, 13, 51] that

$$A_n = (-1)^n \det_{1\le i,j\le n}\big(\{a_{i-j-1}\}\big), \quad C_n = \det_{1\le i,j\le n}\big(\{a_{i-j}\}\big). \tag{2.19}$$

This fact, (2.14), and Jacobi's identity for determinants, knocking off the first and last rows and columns of A_{n+2}, can be used to show that

$$B_n = \det_{1\le i,j\le n}\big(\{a_{i-j-2}\}\big). \tag{2.20}$$

We may expect the other quantities to be similar determinants with one or two modified rows or columns. Indeed, we have some partial results, which we shall give here without derivation. We have found that

$$\begin{aligned} A^+_{n-1} &= (-1)^n \det_{1\le i,j\le n}\big(\{a^+_{i,j}\}\big), \qquad & a^+_{i,j} &= a_{i-j-1}, \text{ except } a^+_{i,1} = p_i\,/k'^3, \\ C^+_n &= \det_{1\le i,j\le n}\big(\{c^+_{i,j}\}\big), & c^+_{i,j} &= a_{i-j}, \text{ except } c^+_{i,1} = p_{i+1}/k'^3, \end{aligned} \tag{2.21}$$

where p_j is determined by the recurrence relation

$$(2n-1)p_{n+2} - \big((2n-4)k + 2n/k\big)p_{n+1} + (2n-3)p_n = 0, \tag{2.22}$$

$$p_1 = -(2k^2-1)a_{-1} - ka_0, \quad p_2 = -ka_{-1} - a_0. \tag{2.23}$$

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*Eqs. (4.16) and (4.17) in [51] have four misprinted signs, as can be easily checked from (4.8), (4.14), and (4.15) there. Also, (4.19) in [51] has $C(n,n)$ and $C^*(n,n)$ interchanged, since A_n is the $T < T_{\mathrm{c}}$ correlation function.

This relation has the very same structure as (2.18), namely a three-term relation with coefficients linear in n and a similar k-dependence. This is not unexpected. In (2.21) we have modified only the first column in both determinants. This is ad hoc, since there are trivially equivalent results moving the column to the very back. Similarly, we have found that

$$\begin{aligned} B_{n-1}^{-} &= (-1)^n \det_{1\le i,j\le n}\big(\{b_{i,j}^{-}\}\big), \qquad b_{i,j}^{-} = a_{i-j-1}, \text{ except } b_{n,j}^{-} = kq_{j+1}/k', \\ D_{n}^{-} &= (-1)^n \det_{1\le i,j\le n}\big(\{d_{i,j}^{-}\}\big), \qquad d_{i,j}^{-} = a_{i-j-1}, \text{ except } d_{1,j}^{-} = q_{j}\ /k', \end{aligned} \tag{2.24}$$

where q_j is determined by the recurrence relation

$$(2n+1)q_{n+2} - \big(2nk + 2n/k\big)q_{n+1} + (2n-1)q_n = 0, \tag{2.25}$$

$$q_1 = a_{-1} + ka_0, \quad q_2 = ka_{-1} + a_0. \tag{2.26}$$

We have some further ideas, but they need to be checked first.

3 Susceptibility Series

With the results of the previous section it is straightforward to compute many terms of the high- and low-temperature series of the susceptibility χ, using the fluctuation-dissipation theorem

$$\bar{\chi} \equiv k_{\mathrm{B}}T\chi = \sum_{m,n=-\infty}^{\infty} \big(\langle\sigma_{0,0}\sigma_{m,n}\rangle - \langle\sigma_{0,0}\rangle^2\big). \tag{3.1}$$

In collaboration with Orrick, Nickel, and Guttmann we have found [54, 55] the high-temperature series of the symmetric square lattice (1.1) to be

$$\begin{aligned} \bar{\chi} = \quad & 1 + 4s + 12s^2 + 32s^3 + 76s^4 + 176s^5 + 400s^6 + \cdots \\ & + 2007330258829172991431166572284107035 66\backslash \\ & \ 232325184536754555022644572376340673830\backslash \\ & \ 1159160108585998318576s^{323}, \end{aligned} \tag{3.2}$$

where $s \equiv \sinh(2K)/2 = \sqrt{k}/2$. The corresponding low-temperature series has been evaluated simultaneously with the result [54, 55]

$$\begin{aligned} \bar{\chi} = \quad & 4s^4 + 16s^6 + 104s^8 + 416s^{10} + 2224s^{12} + \cdots \\ & + 305154772450904435085566207250038946846\backslash \\ & \ 389327390757328102112294342994208496122\backslash \\ & \ 345171749820308452455331887458424846630\backslash \\ & \ 637797467206682914215700492366927125970\backslash \\ & \ 7379855275224873707435550114462001144064s^{646}, \end{aligned} \tag{3.3}$$

where $s \equiv 2/\sinh(2K) = 2/\sqrt{k}$.

The many missing terms "$\cdots$" can be found on Guttmann's webpages [54]. It may seem a little ridiculous to have so many terms with such large numbers for the coefficients. However, it is a well-known fact that the new information in the next coefficient of a series is typically only in the last one or two digits of the number.

As a result, we were able to obtain convincing detailed information about the corrections to scaling of the susceptibility, including an estimate on the accompanying essential singularity [54, 55]. The existence of the essential singularity may also be inferred applying Padé analysis to both the high- and the low-temperature series of the susceptibility. The approximants to the high-temperature series give good results for $T > T_{\rm c}$, but spurious behavior below $T_{\rm c}$. With the low-temperature series it is the other way around. Very recently McCoy [56] has proposed that a similar essential singularity plays a role in hard-sphere gases.

The correction to scaling structure uncovered in [54, 55] has been used recently by Caselle, Hasenbusch, Pelissetto, and Vicari [57] to improve the field-theoretical analysis. As is also noted there, the new features in the corrections to scaling are due to the presence of irrelevant variables. In order to understand these better, we will need additional information moving away from the symmetric square-lattice case. Therefore, we shall continue with the more general Z-invariant inhomogeneous Ising model.

4 Baxter's Z-invariant inhomogeneous Ising model

Commuting transfer matrices [16] leave the partition function Z invariant and imply a great deal of relations among correlation functions. This was first utilized by Baxter who in 1978 introduced the Z-invariant Ising model [5, 16, 51–53, 58–60] as a natural generalization of the uniform Ising model. The Z-invariant model is defined in terms of a set of oriented straight lines carrying "rapidity" variables u_i, v_j, $\cdots$. Only two lines can meet at each intersection and the areas separated by the rapidity lines can be colored alternatingly black and white. An Ising spin is associated with each black area and a dual Ising spin with each white area. Following Baxter [16], the model may be locally deformed to the situation shown in Fig. 1.

In this way we have defined a dual pair of Ising models. Each pair of spins in the first Ising model meets at an intersection of two rapidity lines and has pair interaction $-K\sigma_x\sigma_y$ with $K = \beta J$. Corresponding to this pair of spins, there is a pair of dual spins in the dual Ising model meeting at the same intersection and having interaction $-K^*\sigma_{x^*}\sigma_{y^*}$, where [4]

$$\tanh(K^*) = \exp(-2K), \quad \text{or} \quad \sinh(2K)\sinh(2K^*) = 1. \tag{4.1}$$

In view of the orientation of the rapidity lines, there are two possible choices

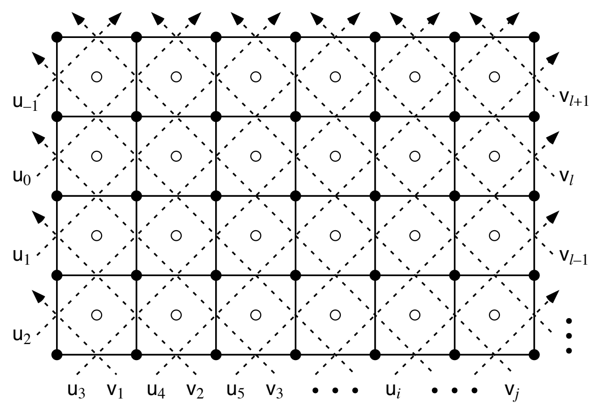

FIGURE 1. Part of the lattice of a two-dimensional Z-invariant Ising model is indicated by black lines. The rapidity lines are represented by oriented dashed lines, carrying rapidity variables u_i and v_j. The positions of the spins are indicated by small black circles, the positions of the dual spins by small white circles.

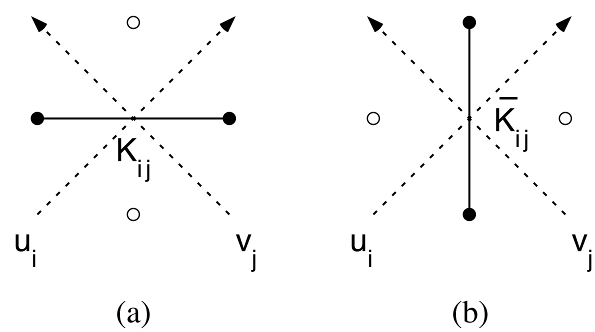

FIGURE 2. The two choices for the coupling constant: K_{ij} and $\bar{K}_{ij}$.

for the reduced interaction strength $K_{x,y}$ of the spins at positions x and y, see Fig. 2. In Fig. 2(a) the two rapidity lines with rapidity variables u_i and v_j pass between the two spins in the same direction and we choose $K_{x,y} = K(u_i, v_j)$. In Fig. 2(b) they pass in opposite directions and in this case we write $K_{x,y} = \bar{K}(u_i, v_j)$.

Following [4, 51, 52, 58], we have a parameterization in terms of elliptic functions of modulus k:

$$\begin{aligned} \sinh\big(2K(u_1,u_2)\big) &= k\,\mathrm{sc}(u_1-u_2,k') = \mathrm{cs}\big(\mathrm{K}(k')+u_2-u_1,k'\big), \quad (4.2)\\ \sinh\big(2\bar{K}(u_1,u_2)\big) &= \mathrm{cs}(u_1-u_2,k') = k\,\mathrm{sc}\big(\mathrm{K}(k')+u_2-u_1,k'\big), \quad (4.3)\end{aligned}$$

where $k' = \sqrt{1-k^2}$ and $\mathrm{sc}(v,k) = \mathrm{sn}(v,k)/\mathrm{cn}(v,k) = 1/\mathrm{cs}(v,k)$. From (4.2) and (4.3) we see that K and $\bar{K}$ are interchanged if we replace u_1 by $u_2 \pm \mathrm{K}(k')$ and u_2 by u_1. This implies that the parameterization is consistent under rotations, as flipping the orientation of a rapidity line j is equivalent to changing its rapidity variable u_j to $u_j \pm \mathrm{K}(k')$.

5 Two-Point Correlation Functions

As pointed out first by Baxter [5], the two-point correlation function in the Z-invariant model can only depend on the elliptic modulus k and the values of the $2m$ rapidity variables $u_1, \ldots, u_{2m}$ that pass between the two spins, implying the existence of an infinite set of universal functions $g_2, g_4, \ldots, g_{2m}, \ldots$ such that for any permutation P and rapidity shift v

$$\langle\sigma\sigma'\rangle = g_{2m}(k; \bar{u}_1, \ldots, \bar{u}_{2m}) = g_{2m}(k; \bar{u}_{\mathrm{P}(1)} + v, \ldots, \bar{u}_{\mathrm{P}(2m)} + v). \tag{5.1}$$

Here we follow [52] and write $\bar{u}_j = u_j$ if the jth rapidity line passes between the two spins σ and σ' in a given direction and $\bar{u}_j = u_j + \mathrm{K}(k')$ if it passes in the opposite direction. This given direction can be chosen for each pair of spins independently in view of the translation symmetry in (5.1). This translation invariance is a direct consequence of the "difference property" of the Boltzmann weights, as follows from (4.2) and (4.3).

If two of the rapidity variables passing between the two spins differ by $\mathrm{K}(k')$, they can be viewed as belonging to a single rapidity line moving back and forth between these two spins. Therefore, we can "move this rapidity line out to infinity" and find with Baxter [5]

$$g_{2m+2}\big(k; \bar{u}_1, \ldots, \bar{u}_{2m}, \bar{u}_{2m+1}, \bar{u}_{2m+1} + \mathrm{K}(k')\big) = g_{2m}(k; \bar{u}_1, \ldots, \bar{u}_{2m}). \tag{5.2}$$

Relations (5.1) and (5.2) are very powerful and can be used in an alternative derivation of the Painlevé equations for the scaling limit [51].

6 Jin's Conjecture of the Scaling Limit of the Two-Point Function

In the critical region we have $k \to 1$ and in this limit the elliptic functions degenerate to trigonometric functions. More precisely, we have

$$\sinh\big(2K(u_1, u_2)\big) = \tan(u_1 - u_2) = \cot(\pm\tfrac{1}{2}\pi + u_2 - u_1), \tag{6.1}$$

$$\sinh\big(2\bar{K}(u_1, u_2)\big) = \cot(u_1 - u_2) = \tan(\pm\tfrac{1}{2}\pi + u_2 - u_1), \tag{6.2}$$

and also $\mathrm{K}(k') = \mathrm{K}(0) = \frac{1}{2}\pi$, which implies that each rapidity variable can be reinterpreted as half the angle of its rapidity line with a fixed direction. Indeed, flipping the orientation changes the angle by $\pm 2\mathrm{K}(0) = \pm\pi$.

Baiqi Jin started out from the known formulae for the two-point function in the scaling limit for the uniform rectangular and triangular Ising lattices, and rewriting them in terms of the proper rapidity variables. The scaling functions for the uniform rectangular Ising lattice were established in a monumental series of papers of Wu, McCoy, Tracy, and Barouch written between 1973 and 1977 [26–29]. For the triangular Ising model Jin used the corresponding results that were established in 1976 by McCoy's student Vaidya [30]. The results in these papers can be rewritten as

$$\langle\sigma\sigma'\rangle \approx |1-k^{-2}|^{1/4}F(r), \quad \langle\sigma\sigma'\rangle^{*} \approx |1-k^{-2}|^{1/4}G(r), \tag{6.3}$$

where the functions $F(r)$ and $G(r)$ are Painlevé functions and the front factor is the square of the spontaneous magnetization for $T < T_{\mathrm{c}}$ or $k > 1$. Furthermore, r is the scaled distance, as both the distance R between the spins and the correlation length ξ become infinite. For this ξ we can take the diagonal correlation length ξ_{d}, which only depends on k [12, 13]. Thus the scaled distance is simply given by

$$r = R/\xi_{\mathrm{d}}, \quad \text{where} \quad \xi_{\mathrm{d}}^{-1} = |\log k|, \tag{6.4}$$

and R is given through [26–30] for the two special uniform lattices.

From the assumption that the scaled correlation function depends on a single distance variable R and from the results for R in the above special cases, Jin made the conjecture that R has the general form

$$R = \frac{1}{2}\biggl[\biggl\{\sum_{j=1}^{2m}\cos(2u_j)\biggr\}^2 + \biggl\{\sum_{j=1}^{2m}\sin(2u_j)\biggr\}^2\biggr]^{1/2}. \tag{6.5}$$

Here the front factor $\frac{1}{2}$ may seem a little bit arbitrary. However, for the special case of diagonal correlations $\langle\sigma_{00}\sigma_{mm}\rangle$ in the uniform rectangular Ising model, for which all $2m$ u_j's are equal, we find then indeed $R = m$, and this normalization is independent of anisotropy (or rapidities).

The choice (6.5) satisfies all the Z-invariance properties of Section 5. As said before, we can view the rapidity variables as angle variables and exploit this fact writing down equivalent unit vectors $\mathbf{e}_j = \bigl(\cos(\lambda u_j), \sin(\lambda u_j)\bigr)$, with $\lambda = 2$. We can then express the required rotation and permutation symmetries (5.1) for R as $R \sim \bigl|\sum \mathbf{e}_j\bigr|$ leading to (6.5). The choice $\lambda = 2$ is also consistent with (5.2), since any pair u and $u + \frac{1}{2}\pi$ must cancel out of the formula for R.[†]

At first, in order to substantiate his conjecture, Jin had verified only that $\langle\sigma\sigma'\rangle \approx A/R^{1/4}$, with universal constant A [13], satisfies the quadratic correlation function relations of [44] at the critical point. Shortly after, we succeeded in generalizing this and showing that the identical Painlevé functions of [26–30] also describe the scaled two-point correlation functions in the general Z-invariant Ising model [51]. We shall explain this in some detail in the next few sections.

[†]A careful reader may note at this point that our arguments only seem to force λ to be an odd multiple of 2. However, if $\lambda \neq \pm 2$, the resulting formula for R implies spurious behavior of the correlation function, inconsistent with a variety of exact results [26–30, 61].

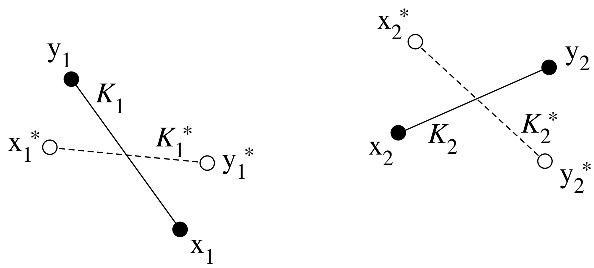

FIGURE 3. Two nearest-neighbor pairs of Ising spins $(\mathrm{x}_1, \mathrm{y}_1)$ and $(\mathrm{x}_2, \mathrm{y}_2)$ on a general planar graph are shown as black circles with reduced interaction constants K_1 and K_2 indicated by drawn lines. The corresponding two pairs of dual spins $(\mathrm{x}_1^*, \mathrm{y}_1^*)$ and $(\mathrm{x}_2^*, \mathrm{y}_2^*)$ are given as white circles with their reduced interaction constants K_1^* and K_2^* indicated by dashed lines.

7 Proof of Jin's Conjecture

In 1980 one of us proved a quadratic identity [44] for the most general planar Ising model relating the two-point correlation function $\langle\sigma_{\mathrm{x}}\sigma_{\mathrm{y}}\rangle$ with its counterpart on the dual lattice $\langle\sigma_{\mathrm{x}^*}\sigma_{\mathrm{y}^*}\rangle^*$, i.e.

$$\begin{aligned}\sinh(2K_1)\sinh(2K_2)\big\{\langle\sigma_{\mathrm{x}_1}\sigma_{\mathrm{x}_2}\rangle\langle\sigma_{\mathrm{y}_1}\sigma_{\mathrm{y}_2}\rangle-\langle\sigma_{\mathrm{x}_1}\sigma_{\mathrm{y}_2}\rangle\langle\sigma_{\mathrm{y}_1}\sigma_{\mathrm{x}_2}\rangle\big\}\\+\big\{\langle\sigma_{\mathrm{x}_1^*}\sigma_{\mathrm{x}_2^*}\rangle^*\langle\sigma_{\mathrm{y}_1^*}\sigma_{\mathrm{y}_2^*}\rangle^*-\langle\sigma_{\mathrm{x}_1^*}\sigma_{\mathrm{y}_2^*}\rangle^*\langle\sigma_{\mathrm{y}_1^*}\sigma_{\mathrm{x}_2^*}\rangle^*\big\}=0,\quad(7.1)\end{aligned}$$

see Fig. 3. Here we have two arbitrary nearest-neighbor pairs of spins at the sites $\{\mathrm{x}_1,\mathrm{y}_1\}\neq\{\mathrm{x}_2,\mathrm{y}_2\}$ with couplings $K_1=\beta J_1$, and $K_2=\beta J_2$, together with the corresponding two nearest-neighbor pairs of dual spins at sites $\{\mathrm{x}_1^*,\mathrm{y}_1^*\}$ and $\{\mathrm{x}_2^*,\mathrm{y}_2^*\}$ with the dual couplings K_1^* and K_2^* satisfying $\sinh(2K_i)\sinh(2K_i^*)=1$, $(i=1,2)$. The orientations of both $(\mathrm{x}_1,\mathrm{x}_1^*,\mathrm{y}_1,\mathrm{y}_1^*)$ and $(\mathrm{x}_2,\mathrm{x}_2^*,\mathrm{y}_2,\mathrm{y}_2^*)$ must be chosen the same, or we must change the plus sign at the beginning of the second line of (7.1) to a minus sign.

Identity (7.1) together with the properties of Section 5 determine the two-point correlation completely as we shall discuss further in Section 11. These should, therefore, suffice to prove Jin's conjecture. So let us restrict ourselves to the Z-invariant Ising model, in which case the quadratic identity (7.1) reduces to

$$\begin{aligned}&k^2\mathrm{sc}(u_2-u_1,k')\mathrm{sc}(u_4-u_3,k')\\&\quad\times\big\{g(u_1,u_2,u_3,u_4,\cdots)g(\cdots)-g(u_1,u_2,\cdots)g(u_3,u_4,\cdots)\big\}\\&+\big\{g^*(u_1,u_3,\cdots)g^*(u_2,u_4,\cdots)-g^*(u_1,u_4,\cdots)g^*(u_2,u_3,\cdots)\big\}=0.\\&\hfill(7.2)\end{aligned}$$

Here the "$\cdots$" is short-hand for all rapidity variables $u_5,u_6,\cdots$ that are common to all the two-point functions g and g^* in (7.2). Eq. (7.2) follows directly if all rapidity lines pass between the spins in the same direction, and more generally with all u_j replaced by $\bar{u}_j$, which means that for those rapidity lines that go in the opposite direction the corresponding u_j are replaced by $u_j + \mathrm{K}(k')$, as in (5.1).

In the scaling limit we have $k \to 1$, $k' \to 0$, and (7.2) reduces to the leading nonvanishing term in the expansion of

$$\tan(u_2 - u_1)\tan(u_4 - u_3)\big\{F(r_{1234})F(r) - F(r_{12})F(r_{34})\big\} \\ + \big\{G(r_{13})G(r_{24}) - G(r_{14})G(r_{23})\big\} = 0. \tag{7.3}$$

Here we are using the notation r for the scaled distance, as given by (6.4) and (6.5), with only the u_k variables common to all the eight two-point functions occurring, i.e. the "$\cdots$" of (7.2). Furthermore, we use r_{ij} with $1 \le i, j \le 4$ for the scaled distance with the variables u_i and u_j added and similarly r_{1234} if u_1, u_2, u_3, u_4 are added. Finally, F and G are the scaling functions corresponding to g and g^*.

More specifically, we can define polar coordinates r and ψ by

$$r\cos\psi = \tfrac{1}{2}\xi_{\mathrm{d}}^{-1} \sum_{j \neq 1,2,3,4} \cos(2u_j), \quad r\sin\psi = \tfrac{1}{2}\xi_{\mathrm{d}}^{-1} \sum_{j \neq 1,2,3,4} \sin(2u_j). \tag{7.4}$$

Since $\xi_{\mathrm{d}} \to \infty$, adding any two or all four of the omitted terms is only an infinitesimally small correction. Therefore, we can expand (7.3) to second order and find

$$\cos(u_1 + u_2 - \psi)\cos(u_3 + u_4 - \psi)(FF'' - F'^2 + r^{-1}GG') \\ + \sin(u_1 + u_2 - \psi)\sin(u_3 + u_4 - \psi)(GG'' - G'^2 + r^{-1}FF') = 0, \tag{7.5}$$

where the primes denote differentiation with respect to r. Since (7.5) must hold for all values of ψ, we are led to

$$FF'' - F'^2 = -r^{-1}GG', \tag{7.6}$$
$$GG'' - G'^2 = -r^{-1}FF'. \tag{7.7}$$

These are the same equations as those for the rotational-invariant scaling functions of the uniform case, as we shall explain in the following sections.

8 Painlevé V Equation

We can take the first derivative of (7.6) and use the resulting equation together with (7.6) and(7.7) to eliminate G' and G''. This way [51], we find

$$G^2 = \frac{-2r^3(FF'' - F'^2)^2}{r^2(FF''' - F'F'') + r(FF'' - F'^2) - FF'}. \tag{8.1}$$

We can substitute the first derivative of (8.1) into the right-hand side of (7.6). Hence, we find a homogeneous cubic equation for $F(r)$ and its first four derivatives, i.e.

$$\begin{aligned}(FF'' - F'^2)(r^4F'''' - 2r^2F'' + rF') + FF'^2 \\ +r^4(2F'F''F''' - FF'''^2 - F''^3) = 0.\end{aligned} \tag{8.2}$$

Clearly, both $F(r)$ and $G(r)$ satisfy the same equation (8.2), which is an equation for the tau-function of a special Painlevé V equation. In order to show this, we need to introduce a new dependent variable [38–40]

$$\zeta = rF'/F, \qquad \text{or alternatively } \zeta = rG'/G, \tag{8.3}$$

changing (8.2) into

$$\begin{aligned}r^3(\zeta'\zeta''' - \zeta''^2) - r^2(\zeta\zeta''' - \zeta'\zeta'') - r\zeta\zeta'' \\ +\zeta\zeta' + 2r^2\zeta'^3 - 6r\zeta\zeta'^2 + 4\zeta^2\zeta' = 0.\end{aligned} \tag{8.4}$$

Equation (8.4) is the derivative of

$$\big(r^2\zeta''^2 + 4\zeta'^2(r\zeta' - \zeta) - \zeta'^2\big)/\big(4(r\zeta' - \zeta)^2\big) = \mu^2 \tag{8.5}$$

where μ is a constant setting the "mass scale." Hence, we have shown that the same Painlevé V equation,

$$(r\zeta'')^2 = 4\mu^2(r\zeta' - \zeta)^2 - 4\zeta'^2(r\zeta' - \zeta) + \zeta'^2 \tag{8.6}$$

and its derivative

$$r^2\zeta''' + r\zeta'' = 4\mu^2 r(r\zeta' - \zeta) - 4\zeta'(r\zeta' - \zeta) - 2r\zeta'^2 + \zeta', \tag{8.7}$$

discovered by Jimbo and Miwa for the uniform rectangular Ising model [38,40], applies to the scaled two-point functions of the general Z-invariant Ising model. Comparing with the special uniform rectangular case, we see that we may choose $\mu = 1$. We note that the conformal limit $\mu = 0$ has also been discussed in some detail in [40].

The Painlevé V formulation is very powerful and can be used to expand $F(r)$ and $G(r)$ iteratively to many orders using Maple, for example. We have gone to 154 orders on an old Macintosh computer and our result has the form

$$\begin{aligned}F(r) &\approx \frac{\tilde{A}}{r^{1/4}}\bigg\{1 + \sum_{n=1}^{154}\sum_{m=0}^{\lfloor\sqrt{n}\rfloor} c_{m,n}\big(\ln(r/8) + \gamma_{\mathrm{E}}\big)^m r^n\bigg\}, \\ G(r) &\approx \frac{\tilde{A}}{r^{1/4}}\bigg\{1 + \sum_{n=1}^{154}\sum_{m=0}^{\lfloor\sqrt{n}\rfloor} (-1)^n c_{m,n}\big(\ln(r/8) + \gamma_{\mathrm{E}}\big)^m r^n\bigg\},\end{aligned} \tag{8.8}$$

where γ_{E} is Euler's constant and $\tilde{A} = \mathrm{e}^{3\zeta'(-1)}2^{-1/6}$ includes a factor $2^{-1/4}$ coming from $\xi_{\mathrm{d}}^{-1/4}/(1-k^{-2})^{1/4}$ in the limit $k \to 1$. We have used the initial values $c_{0,1} = 0, c_{1,1} = \frac{1}{2}, c_{0,2} = \frac{1}{16}, c_{1,2} = 0$ to start the iteration process. The first four orders reproduce those of [28] and the decay of the coefficients indicates that the expansions (8.8) converge for all positive values of r. Note that the maximal number of logarithmic factors per term in (8.8) is at most the square root of the order.

The first terms in the large-r asymptotic expansions of $F(r)$ and $G(r)$ are explicitly known in terms of modified Bessel functions [28], namely

$$\begin{aligned} F_{\mathrm{asy}}(r) &= \frac{1}{\pi} K_0(r), \\ G_{\mathrm{asy}}(r) &= 1 + \frac{1}{\pi^2}\big[r^2\big(K_1(r)^2 - K_0(r)^2\big) - rK_0(r)K_1(r) + \frac{1}{2}K_0(r)^2\big]. \end{aligned} \tag{8.9}$$

The corrections are only $\mathrm{O}(\mathrm{e}^{-3r})$ and $\mathrm{O}(\mathrm{e}^{-4r})$, respectively. Unfortunately, no such expressions are available for the next order in these asymptotic expansions. However, the agreement of the small-r expansions (8.8) and the asymptotic formulae (8.9) is remarkable. Using (8.8) for $r < 11.4$ and (8.9) for $r > 11.4$ determines $F(r)$ to better accuracy than 5×10^{-24}. Using crossover point $r = 10.2$ the error in $G(r)$ is even less than 2×10^{-31}! This can be used in an alternative method to accurately determine the amplitudes of the susceptibility singularity and the results agree with those of Nickel [36, 54].

For most purposes it is sufficient to have 10 or less terms in (8.8), i.e.

$$\begin{aligned} F(r) \approx \frac{\tilde{A}}{r^{1/4}} \bigg\{ & 1 + \frac{1}{2}\,Lr + \frac{1}{16}\,r^2 + \frac{1}{32}\,Lr^3 \\ & + \Big(-\frac{1}{256}\,L^2 + \frac{1}{256}\,L + \frac{1}{2048}\Big) r^4 + \Big(\frac{5}{4096}\,L - \frac{1}{4096}\Big) r^5 \\ & + \Big(-\frac{1}{4096}\,L^2 + \frac{1}{4096}\,L - \frac{5}{98304}\Big) r^6 + \Big(\frac{7}{196608}\,L - \frac{1}{65536}\Big) r^7 \\ & + \Big(-\frac{17}{2097152}\,L^2 + \frac{35}{4194304}\,L - \frac{469}{201326592}\Big) r^8 \\ & + \Big(-\frac{1}{8388608}\,L^3 + \frac{5}{16777216}\,L^2 + \frac{209}{402653184}\,L - \frac{209}{536870912}\Big) r^9 \\ & + \Big(-\frac{19}{100663296}\,L^2 + \frac{41}{201326592}\,L - \frac{937}{16106127360}\Big) r^{10} \bigg\}, \end{aligned} \tag{8.10}$$

where

$$L \equiv \ln(r) - \ln(8) + \gamma_{\mathrm{E}}. \tag{8.11}$$

This gives $F(r)$ with an error smaller than a few times 10^{-8} if one uses $F_{\mathrm{asy}}(r)$ for $r > 1.41$. Replacing r by $-r$ within the curly brackets of (8.10) changes $F(r)$ to $G(r)$, which is equally accurate if one takes $G_{\mathrm{asy}}(r)$ for $r > 0.83$.

Originally [26–29], the scaling functions $F(r)$ and $G(r)$ were given in terms of a Painlevé III formulation. The Painlevé V language of [38–40] has not been applied much to the two-dimensional Ising model. It has been very successful in the one-dimensional XY-model [62, 63]. We shall discuss the relationship between the two approaches further in the next section.

9 Sinh-Gordon and Painlevé III Equation

One discussion of the equivalence of the Painlevé III and V formulations has been given in [40]. This heavily relies on several earlier works. There is, however, a more direct derivation starting from the scaling limit of (2.1) through (2.4), namely

$$(FF_{xx} - {F_x}^2) + (GG_{yy} - {G_y}^2) = 0, \tag{9.1}$$

$$(FF_{yy} - {F_y}^2) + (GG_{xx} - {G_x}^2) = 0, \tag{9.2}$$

$$(FF_{xy} - F_xF_y) - (GG_{xy} - G_xG_y) = 0, \tag{9.3}$$

$$(FG_{xx} - 2F_xG_x + F_{xx}G) + (FG_{yy} - 2F_yG_y + F_{yy}G) = \lambda FG, \tag{9.4}$$

which follows writing $m = \xi x$ and $n = \xi y$, for some choice of $\xi \to \infty$. Here

$$\lambda = 2(\sqrt{k} - 1)^2\xi^2/\sqrt{k} \tag{9.5}$$

is a nonnegative mass parameter that can be chosen arbitrarily in the limit $k \to 1$, $\xi \propto \xi_{\mathrm{d}}$ and the subscripts x and y denote partial differentiations, which we shall also write as ∂_x and ∂_y.

Equation (9.4) is an integral of the other three, as we shall explain below, and (9.3) is more or less implying the rotational invariance of $F = F(r)$ and $G = G(r)$, in terms of the usual polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$. Assuming this, (9.1) and (9.2) reduce immediately to (7.6) and (7.7) and the resulting Painlevé V formalism of Section 8.

In order to get to the sinh-Gordon equation and from that the Painlevé III formulation of [26–29], it is expedient to introduce the product and quotient variables

$$P \equiv FG, \qquad Q \equiv F/G, \tag{9.6}$$

where Q is the same as the function G in the original literature [26–29]. Then (9.1) through (9.4) become

$$Q^2{\partial_x}^2\log(PQ)+{\partial_y}^2\log(P/Q)=0, \tag{9.7}$$

$$Q^2{\partial_y}^2\log(PQ)+{\partial_x}^2\log(P/Q)=0, \tag{9.8}$$

$$Q^2\partial_x\partial_y\log(PQ)-\partial_x\partial_y\log(P/Q)=0, \tag{9.9}$$

$$({\partial_x}^2+{\partial_y}^2)\log P+(\partial_x\log Q)^2+(\partial_y\log Q)^2=\lambda. \tag{9.10}$$

From the first three of these we can solve

$${\partial_x}^2\log P=\frac{(1+Q^4){\partial_x}^2\log Q-2Q^2{\partial_y}^2\log Q}{1-Q^4}, \tag{9.11}$$

$${\partial_y}^2\log P=\frac{(1+Q^4){\partial_y}^2\log Q-2Q^2{\partial_x}^2\log Q}{1-Q^4}, \tag{9.12}$$

$$\partial_x\partial_y\log P=\frac{1+Q^2}{1-Q^2}\,\partial_x\partial_y\log Q, \tag{9.13}$$

which we can substitute in (9.10). This results in

$$\frac{1-Q^2}{1+Q^2}\,({\partial_x}^2+{\partial_y}^2)\log Q+(\partial_x\log Q)^2+(\partial_y\log Q)^2=\lambda. \tag{9.14}$$

This is a well-known variant of the Euclidian sinh-Gordon equation [28], as can be seen substituting

$$Q=\tanh(\tfrac{1}{2}\phi), \tag{9.15}$$

so that

$$({\partial_x}^2+{\partial_y}^2)\phi=\tfrac{1}{2}\lambda\sinh(2\phi). \tag{9.16}$$

Setting $\lambda=1$ by proper choice of ξ and using rotational invariance this becomes the Painlevé III equation [28]

$$\frac{\mathrm{d}^2\eta}{\mathrm{d}r^2}-\frac{1}{\eta}\Big(\frac{\mathrm{d}\eta}{\mathrm{d}r}\Big)^2+\frac{1}{r}\frac{\mathrm{d}\eta}{\mathrm{d}r}+\frac{1-\eta^4}{4\eta}=0,\qquad \eta(r)\equiv\mathrm{e}^{\pm\phi}, \tag{9.17}$$

where one can choose either sign.

Calling the left-hand side of (9.14) X and using (9.11), (9.12), and (9.13), we find that

$$\partial_xX=\frac{1-Q^2}{2Q^2}\big(\partial_y(\partial_x\partial_y\log P)-\partial_x({\partial_y}^2\log P)\big)=0,$$

$$\partial_y X = \frac{1-Q^2}{2Q^2}\big(\partial_x(\partial_x\partial_y \log P) - \partial_y({\partial_x}^2 \log P)\big) = 0, \qquad (9.18)$$

showing that X is a constant and that (9.4) is indeed an integral of the other three equations. Therefore, in view of the rotational invariance of the scaling limit, (7.6) and (7.7) do indeed fully determine the scaling functions $F(r)$ and $G(r)$.

10 Susceptibility in Z-Invariant Lattice

For general ferromagnetic Z-invariant lattice with $\mathcal{N}$ sites, the susceptibility χ is given by

$$\bar{\chi} \equiv k_{\mathrm{B}}T\chi = \lim_{\mathcal{N}\to\infty}\frac{1}{\mathcal{N}}\sum_{m_1,n_1}\sum_{m_2,n_2}\big(\langle\sigma_{m_1,n_1}\sigma_{m_2,n_2}\rangle - \langle\sigma_{0,0}\rangle^2\big), \qquad (10.1)$$

where (m_1, n_1) and (m_2, n_2) run through the possible coordinates of the spins in some suitable system. In periodic cases one of the two sums can be done trivially. In quasiperiodic cases this can only be done asymptotically at the largest distance scale. Hence, in the scaling limit and for both periodic and quasiperiodic Z-invariant lattices, (10.1) becomes

$$\bar{\chi} \approx g_0 \int_{-\infty}^{+\infty}\mathrm{d}M \int_{-\infty}^{+\infty}\mathrm{d}N\, \frac{\check{F}_{\pm}(\kappa R)}{R^{1/4}}, \qquad (10.2)$$

where[‡]

$$\frac{\check{F}_{-}(\kappa R)}{R^{1/4}} = |1-k^{-2}|^{1/4}\big(G(R/\xi_{\mathrm{d}})-1\big),$$
$$\frac{\check{F}_{+}(\kappa R)}{R^{1/4}} = |1-k^{-2}|^{1/4}\, F(R/\xi_{\mathrm{d}}), \qquad (10.3)$$

$\kappa = 1/\xi_{\mathrm{d}} = |\log k|$, and R given in (6.5) reduces to

$$R = \sqrt{aM^2 + 2bMN + cN^2}, \qquad (10.4)$$

with a, b, and c known constants that can be calculated from (6.5) choosing suitable integer coordinates M and N. In addition, g_0 is the corresponding multiplicity factor counting how many spin distance vectors fall exactly or asymptotically within a unit cell in the (M, N) plane. Therefore,

$$\bar{\chi} = \frac{2\pi g_0}{\sqrt{ac-b^2}} \int_0^{\infty}\mathrm{d}r\, r^{3/4}\check{F}_{\pm}(r)\,\kappa^{-7/4} = A_{\pm}|t|^{-7/4}, \qquad (10.5)$$

[‡]Note that our $\check{F}_{\pm}(r)$ differs by a constant factor ζ, $\check{F}_{\pm}(r) = \zeta F_{\pm}(r)$, compared to $F_{\pm}(r)$ defined in [28], since we chose a different definition of R to account for Z-invariance. In the symmetric square lattice case $\zeta = 2^{1/8}$.

with $t \equiv |T - T_{\rm c}|/T_{\rm c}$, giving the exact $T \gtrless T_{\rm c}$ susceptibility amplitudes for all (quasi)periodic Z-invariant lattices. Note that (10.5) implies that the ratio A_+/A_- is universal for all (quasi)periodic ferromagnetic Z-invariant Ising models.

For the analysis of the long susceptibility series in the isotropic square lattice the values of constants proportional to A_+ and A_- were evaluated to very high precision by Nickel [54]. From these values we can obtain A_+ and A_- for the isotropic square (sq), triangular (tr) and honeycomb (hc) lattices using (10.5), i.e.

$$\begin{aligned}
A_+^{\rm sq} &= 0.962581732308772114044329809433469495167139194757936 5,\\
A_+^{\rm tr} &= 0.924206958245164329697157577855931717669626152002838 9,\\
A_+^{\rm hc} &= 1.046417076152338359733871672674357433252295746539088,\\
A_-^{\rm sq} &= 0.025536974522023905385953456226398471929219687270774 55,\\
A_-^{\rm tr} &= 0.024518904477000004890808552397197720236530228514229 50,\\
A_-^{\rm hc} &= 0.027761098425397045077433797952582855036099698776332 51.
\end{aligned} \tag{10.6}$$

These values may be needed later to analyze the long series for the isotropic triangular and honeycomb lattices, once they are available. We note that these numbers agree with the series extrapolations of [64] and the four earlier values of [26–30]. It is remarkable that they agree to better than three places with those obtained from the Syozi-Naya approximation [8] for T above $T_{\rm c}$, but this can be understood as this approximation is precisely the $\chi_<^{(1)}$ approximation in [28].

In the derivation of (10.6) we have used the following: For the square lattice we have used the standard coordinates (M, N) leading to $a = c = 1/2$, $b = 0$, $g_0 = 1$, and also ${\rm d}k/{\rm d}T = 4\sqrt{2}/T_{\rm c}^2$ at $T_{\rm c} = 2/\log(\sqrt{2}+1)$ in units of $J/k_{\rm B}$. For the isotropic triangular lattice we have taken the coordinates of the equivalent square lattice with one set of diagonals, leading to $a = c = 3/4$, $b = -3/8$, $g_0 = 1$, and now ${\rm d}k/{\rm d}T = 8/T_{\rm c}^2$ at $T_{\rm c} = 4/\log 3$. For the isotropic honeycomb lattice we have used the dual of the same lattice, leading again to $a = c = 3/4$, $b = -3/8$, but $g_0 = 2$, and here ${\rm d}k/{\rm d}T = 8\sqrt{3}/(3T_{\rm c}^2)$ at $T_{\rm c} = 2/\log(2+\sqrt{3})$.

We can also find the more general formulae for the wavevector-dependent susceptibility $\chi(q_x, q_y)$ in the scaling limit, generalizing both (10.5) and the results of [19, 27, 65, 66] to more general Z-invariant cases. We shall not present these results here, as they can be obtained straightforwardly.

11 Sufficiency of Z-Invariance and the Quadratic Difference Equations

In this section, as announced before, we shall discuss how the quadratic identity (7.1) and Z-invariance together determine all two-point correlation functions. In Section 7 it has already been shown that these two principles lead to quadratic identity (7.2). All what is needed more is the dual of this identity, namely

$$\begin{aligned}&\mathrm{sc}(u_2-u_1,k')\mathrm{sc}(u_4-u_3,k')\\&\quad\times\big\{g^*(u_1,u_2,u_3,u_4,\cdots)g^*(\cdots)-g^*(u_1,u_2,\cdots)g^*(u_3,u_4,\cdots)\big\}\\&+\big\{g(u_1,u_3,\cdots)g(u_2,u_4,\cdots)-g(u_1,u_4,\cdots)g(u_2,u_3,\cdots)\big\}=0.\end{aligned}\tag{11.1}$$

From (7.2) and (11.1) it is obvious how to calculate $g(u_1,u_2,u_3,u_4,\cdots)$ and $g^*(u_1,u_2,u_3,u_4,\cdots)$ in terms of correlation functions with two or four less rapidity variables. Since we know the correlation functions with zero or two rapidity variables, i.e. $g(\emptyset)=g^*(\emptyset)=1$ and both $g(u_1,u_2)$ and $g^*(u_1,u_2)$ are known from Onsager's free energy solution [4, 13, 58], we can calculate all other correlation functions iteratively.

There is one complication: If all rapidity variables—or all but one—are equal, (7.2) and (11.1) become trivial identities, unless we resort to l'Hospital's rule. This explains the occurrence of Wronskian determinants in [52,67]. We shall illustrate how this works with a small Maple™ routine, calculating the diagonal and next-to-the-diagonal correlation functions in the uniform square lattice at the critical point, where $g\equiv g^*$.

Example 11.1. Maple routine for critical square lattice

In this example, we use the notations $g[i]=C(i,i)$ and $f[i]=C(i-1,i)$ for $i=1,\cdots,6$. In the critical case $k=1,k'=0$, (7.2) and (11.1) collapse to one equation and $\mathrm{sc}(u,k')=\tan(u)$. We give our Maple program with its output following:

```
h:=1: g[0]:=h: print([0,`--`,g[0]]);
z:=2*(u-v)/Pi/sin(u-v):
z1:=2*u/Pi/sin(u): z2:=subs(u=v,z1):
f[1]:=subs(u=Pi/4,z1):
z0:=2/Pi: g[1]:=z0: print([1,f[1],g[1]]);
for i from 2 to 6 do
  z:=z1*z2/h-(z*z0-z1*z2)/h*cot(u)*cot(v):
  h:=z0: z1:=limit(z,v=0): z2:=subs(u=v,z1):
  f[i]:=subs(u=Pi/4,z1);
  z0:=limit(z1,u=0): g[i]:=z0;
```

```
print([i,f[i],g[i]]); od:
```

$$[0, \text{—}, 1]$$

$$[1, \frac{1}{2}\sqrt{2}, \frac{2}{\pi}]$$

$$[2, 4\,\frac{\sqrt{2}}{\pi^2}, \frac{16}{3}\,\frac{1}{\pi^2}]$$

$$[3, \frac{32}{9}\,\frac{\sqrt{2}}{\pi^2}, \frac{2048}{135}\,\frac{1}{\pi^3}]$$

$$[4, \frac{65536}{2025}\,\frac{\sqrt{2}}{\pi^4}, \frac{1048576}{23625}\,\frac{1}{\pi^4}]$$

$$[5, \frac{8388608}{275625}\,\frac{\sqrt{2}}{\pi^4}, \frac{34359738368}{260465625}\,\frac{1}{\pi^5}]$$

$$[6, \frac{70368744177664}{246140015625}\,\frac{\sqrt{2}}{\pi^6}, \frac{4503599627370496}{11371668721875}\,\frac{1}{\pi^6}]$$

These results agree of course with our earlier results [61]. It is, however, much more efficient to calculate those numbers following the procedure in [61]. For $k \neq 1$ we can use the method of Jimbo and Miwa described in Section 2. However, since no such procedure exists for either the next-to-the-diagonal or for the row correlation functions, our best method is still to use Toeplitz determinants in all cases but the symmetric square lattice [52,68]. From a fundamental point of view, it is clearly desirable to generalize the results of Jimbo and Miwa [38].

12 Conclusions and Outlook

In the previous several sections we have reviewed the current status of calculating two-spin correlation functions and the magnetic susceptibility in the two-dimensional Ising model in zero magnetic field. We have also presented some recent developments using Baxter's Z-invariance, providing both simplifications in the derivations and generalizations of the results.

In the near future, we hope to use these results to extend the high- and low-temperature series for the susceptibility in triangular and honeycomb Ising lattices [64] to many more orders, which should provide valuable new information on the effect of irrelevant variables on corrections to scaling, not available from [54,55,57]. We have made the first few steps on this project, also using ideas on transformations between models from [18,69,70].

With the above information, we should be able to extend our previous work on quasiperiodic Ising models [51]. One such extension is published elsewhere in these proceedings [68].

A much more difficult project is to upgrade the work presented here to four-point functions. In principle, quadratic difference equations are also known for this case, written down explicitly in [47]. The analysis will be much more involved, but could give new information on the magnetic-field dependence.

An even more ambitious project would be to upgrade the current work to non-Ising Z-invariant models. A lot of work has been done the last ten years [71], but the formalism developed is much more complicated and many breakthroughs are still needed to be made.

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